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Reviewed work(s):
Source: *Econometric Theory*, Vol. 14, No. 6 (Dec., 1998), pp. 701-743
Published by: Cambridge University Press
Stable URL: http://www.jstor.org/stable/3533088
Accessed: 20/07/2012 11:08

Econometric Theory, **14**, 1998, 701–743. Printed in the United States of America.

BAYESIAN SIMULTANEOUS EQUATIONS ANALYSIS USING REDUCED RANK STRUCTURES

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Diffuse priors lead to pathological posterior behavior when used in Bayesian analyses of simultaneous equation models (SEM's). This results from the local nonidentification of certain parameters in SEM's. When this a priori known feature is not captured appropriately, it results in an a posteriori favoring of certain specific parameter values that is not the consequence of strong data information but of local nonidentification. We show that a proper consistent Bayesian analysis of a SEM explicitly has to consider the reduced form of the SEM as a standard linear model on which nonlinear (reduced rank) restrictions are imposed, which result from a singular value decomposition. The priors/posteriors of the parameters of the SEM are therefore proportional to the priors/posteriors of the parameters of the linear model under the condition that the restrictions hold. This leads to a framework for constructing priors and posteriors for the parameters of SEM's. The framework is used to construct priors and posteriors for one, two, and three structural equation SEM's. These examples together with a theorem, showing that the reduced forms of SEM's accord with sets of reduced rank restrictions on standard linear models, show how Bayesian analyses of generally specified SEM's can be conducted.

1. INTRODUCTION

Since the early 1940's a lot of research has focused on the development of statistical methods for analyzing simultaneous equation models (SEM's) (see, e.g., Haavelmo, 1943; Anderson and Rubin, 1949). It shows that models that are able to generate variables simultaneously are important because this is a stylized fact of many economic time series. The SEM is not only important but also rather complicated as a result of the problems regarding the identification of its parameters. The identification of the structural parameters is reflected in the rank and order con-

Part of this research was carried out while the first author was hosted at the department of Econometrics at Tilburg University and during his PPS fellowship of the Netherlands Foundation for Scientific Research (N.W.O). We thank four anonymous referees, Luc Bauwens, Peter Phillips, Dale Poirier, Arnold Zellner, Eric Zivot, and other seminar participants for helpful comments and suggestions. This paper has previously circulated as "Bayesian Simultaneous Equations Analysis: On the Existence of Structural Posterior Moments" and "Bayesian Simultaneous Equations Analysis Using Noninformative Priors" (see Kleibergen and van Dijk, 1992, 1994a). Address correspondence to: Frank Kleibergen, Econometric Institute, Erasmus University Rotterdam, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands.

 0266-4666/98 $9.50

ditions that result from the implied reduced form (see Hausman, 1983). The order condition reflects overall identification whereas the rank condition reflects local (non) identification. This latter phenomenon, local nonidentification, is shown to lead to pathological posterior behavior when flat priors are used in Bayesian analyses of the SEM. This behavior occurs in the traditional Bayesian analyses of SEM's documented in the literature (see, e.g., Drèze, 1976; Drèze and Morales, 1976; Drèze and Richard, 1983). We show its occurrence in a limited information (one equation) analysis of the SEM. Similar behavior can be found in other specifications of the SEM as well because the origin of the pathological posterior behavior, local nonidentification of parameters, is exemplary to SEM's.

To obtain a consistent Bayesian analysis of a SEM that does not suffer from these pathologies, we construct a framework in which the reduced form of a SEM is specified as a multivariate linear model with nonlinear (reduced rank) restrictions on its parameters. Using singular value decompositions we specify the restrictions such that a one-to-one correspondence with a linear model is obtained when the restrictions do not hold and the reduced form of the SEM is obtained when they hold. The prior and posterior analysis then results when this specification is used in the framework for analyzing nested models as parameter restrictions of embedding models constructed in Kleibergen (1997). It a.o. leads to invariance of the priors and posteriors with respect to the specification of the model. The resulting posteriors of the parameters of the SEM accord with the posterior of the embedding linear model. Our analysis is therefore similar to the construction of the Savage–Dickey density ratio (see Dickey, 1971). That is, we construct the priors/posteriors in the points where the hypothesis (restriction) holds. In contrast, the posterior of the parameters of a SEM, derived in the usual way using a diffuse prior, is inconsistent in the sense that its implied posterior of the parameters of the embedding linear model is not a member of the standard class of posteriors of the parameters of linear models (see Kleibergen, 1997).

The paper is organized as follows. In Section 2, we show the pathologies arising in the posteriors of the parameters of an incomplete (one structural equation analysis of a) SEM when flat priors are used. Sections 3 and 4 show how an incomplete SEM is rewritten as a multivariate linear model with nonlinear parameter restrictions. We use this specification jointly with the framework for analyzing nested models as parameter restrictions of embedding models to obtain the prior and posterior analysis. Singular value decompositions are also involved that are similar to the canonical correlations used in a limited information maximum likelihood analysis (see Anderson and Rubin, 1949). In Section 5, posterior simulators are constructed to sample from the posterior of the parameters of an incomplete SEM. Section 6 extends the one structural equation analysis to a full system analysis by showing that a fully specified SEM accords with a set of reduced rank restrictions on a linear model. Different subsections then show the framework for prior and posterior analysis for two and three structural equations and also show that the order condition for a full system analysis of a SEM can differ from the order condition resulting from a one structural equation analysis. Finally, Section 7 contains conclusions.

2. NONIDENTIFICATION AND PATHOLOGICAL POSTERIOR BEHAVIOR

To show the consequences that local nonidentification of parameters of SEM's has for posterior distributions, we analyze, as an example, the case of one (set of) structural equation(s). This model is also known as the incomplete simultaneous equations model (INSEM). As the results for the posteriors of the INSEM are exemplary for other specifications of the SEM, the importance of a proper treatment of the issue of local nonidentification is shown by the analysis of the INSEM.

We use as specification of the INSEM (see Zellner, Bauwens, and van Dijk, 1988)

$$\begin{aligned} y_1 &= Y_2\beta + Z_1\gamma + \varepsilon_1, \\ Y_2 &= Z_1\Pi_{12} + Z_2\Pi_{22} + \varepsilon_2, \end{aligned} \tag{2.1}$$

where $y_1 : T \times 1$ and $Y_2 : T \times (m-1)$ are endogenous and $Z_1 : T \times k_1$ and $Z_2 : T \times k_2$, $k = k_1 + k_2$ contain the (weakly) exogenous (see Engle, Hendry, and Richard, 1983), and lagged dependent variables, $\beta : (m-1) \times 1$, $\gamma : k_1 \times 1$, $\Pi_2 = (\Pi_{12}' \ \Pi_{22}')' : k \times (m-1)$ and we assume that $(\varepsilon_1 \ \ \varepsilon_2) \sim n(0, \Sigma \otimes I_T)$. The identification problems arise when the parameter $\Pi_{22} = 0$ (or has reduced rank) as (parts of) the structural form parameter β is then nonidentified. This is easily seen when we construct the reduced form of the INSEM (2.1),

$$\begin{aligned} y_1 &= Z_1\pi_{11} + Z_2\Pi_{22}\beta + \xi_1, \\ Y_2 &= Z_1\Pi_{12} + Z_2\Pi_{22} + \varepsilon_2, \end{aligned} \tag{2.2}$$

where $\pi_{11} = \gamma + \Pi_{12}\beta$, $\xi_1 = \varepsilon_1 + \varepsilon_2\beta$, $(\xi_1 \ \ \varepsilon_2) \sim n(0, \Omega)$, $\Sigma = B'\Omega B$, and

$$B = \begin{pmatrix} 1 & 0 \\ -\beta & I_{m-1} \end{pmatrix}.$$

When $\Pi_{22} = 0$, β is not identified in (2.2) and the disturbances ξ_1 are not affected by the value of β. So, the likelihood is flat and nonzero in the direction of β when $\Pi_{22} = 0$. If we use flat (diffuse) priors in a Bayesian analysis of the INSEM, such that the joint posterior is proportional to the likelihood, the joint posterior of the different parameters will also be flat and nonzero in the direction of β for zero values of Π_{22}. This property is passed on to the marginal posteriors, which are the integrals of the joint posterior over the different parameters. To show the consequences for the marginal posteriors in practice, we calculated the marginal posteriors of the parameters of the demand equation of the "Tintner meat market" model (see Tintner, 1952). In this exact identified model, y_1 reflects quantity of meat consumed, y_2 is the price of meat, z_1 is national income per capita, z_2 is the cost of processing meat (all variables are in deviation from their mean), $m = 2$, $k_1 = k_2 = 1$.

In Figure 1, the joint posterior of β and Π_{22} is drawn for the Tintner meat market dataset, and Figure 2 contains the contour lines of this bivariate posterior. The functional form of this posterior is obtained by using a flat prior ($\propto 1$) and integrating out $(\Sigma, \pi_{11}, \Pi_{12})$ and reads

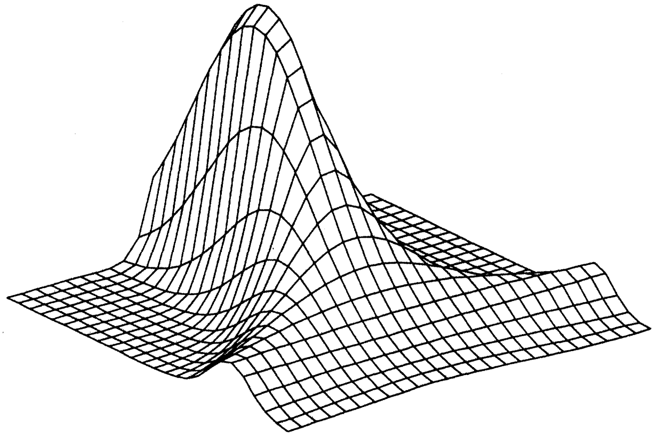

FIGURE 1. Bivariate posterior (β, π_{22}) demand equation Tintner model.

$$p(\beta, \Pi_{22} | Y, Z) \propto |(y_1 - Z_2\Pi_{22}\beta)' M_{(Z_1 \varepsilon_2)} (y_1 - Z_2\Pi_{22}\beta)|^{-(1/2)(T-k_1-m-1)}$$
$$\times |(Y_2 - Z_2\Pi_{22})' M_{Z_1} (Y_2 - Z_2\Pi_{22})|^{-(1/2)(T-k_1-m-1)}, \qquad \textbf{(2.3)}$$

as $Y_2 = Z_1\Pi_{12} + Z_2\Pi_{22} + \varepsilon_2$ and $M_V = I_T - V(V'V)^{-1}V'$, $V = Z_1$, $V = (Z_1 \;\; \varepsilon_2)$. Both Figures 1 and 2 and the functional form of the posterior in (2.3) show that

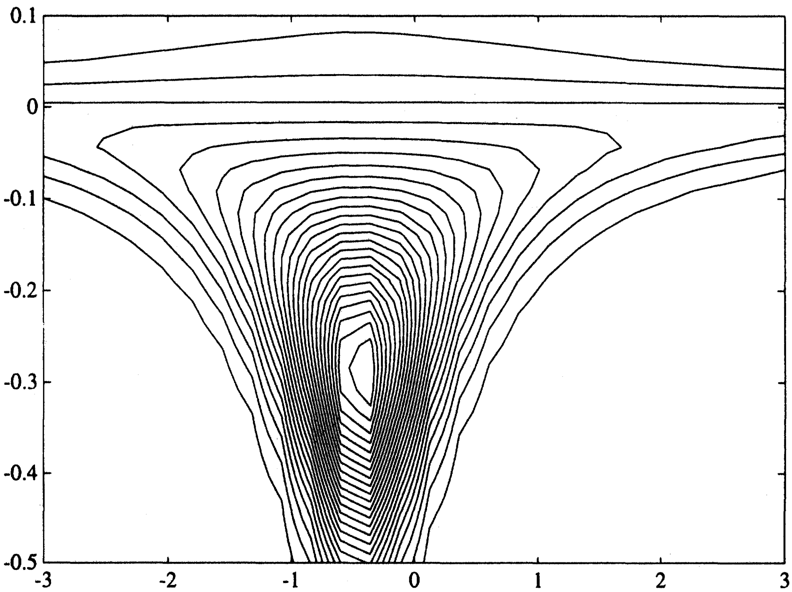

FIGURE 2. Contour lines marginal posterior (β, Π_{22}) demand equation.

the marginal posterior does not depend on β when $\Pi_{22} = 0$ as it is flat and nonzero in the direction of β for zero values of Π_{22}. This implies that the marginal posterior of Π_{22}, which is the integral of the posterior (2.3) over β, will be infinite at $\Pi_{22} = 0$ as at this particular value of Π_{22} we construct an integral of a function over an infinite parameter region whereas the function itself does not depend on the parameter β over which we integrate. So, the integral will be proportional to the size of the parameter region, i.e., infinity. Both the functional form of the marginal posterior of Π_{22},

$$p(\Pi_{22}|Y,Z) \propto |\Pi_{22}' Z_2' M_{(Z_1 \varepsilon_2)} Z_2 \Pi_{22}|^{-1/2} \times \left[\frac{|\Pi_{22}' Z_2' M_{(Z_1\, Y_2)} Z_2 \Pi_{22}|}{|\Pi_{22}' Z_2' M_{(Z_1\, Y_1\, Y_2)} Z_2 \Pi_{22}|} \right]^{-(1/2)(T-k_1-2(m-1))} \times |(Y_2 - Z_2\Pi_{22})' M_{Z_1} (Y_2 - Z_2\Pi_{22})|^{-(1/2)(T-k_1-m-1)}, \tag{2.4}$$

and the marginal posterior of Π_{22} for the Tintner dataset from Figure 3 show this phenomenon, and consequently the value of the posterior of Π_{22} is infinite at $\Pi_{22} = 0$.

The nonidentification of β also has consequences for its own marginal posterior, which belongs to the class of $1 - 1$ poly t densities. See Bauwens and van Dijk (1989), Drèze (1976), Drèze (1977), Drèze and Richard (1983), and Richard and Tompa (1980) for an efficient algorithm to calculate the moments of this class of densities. This posterior reads

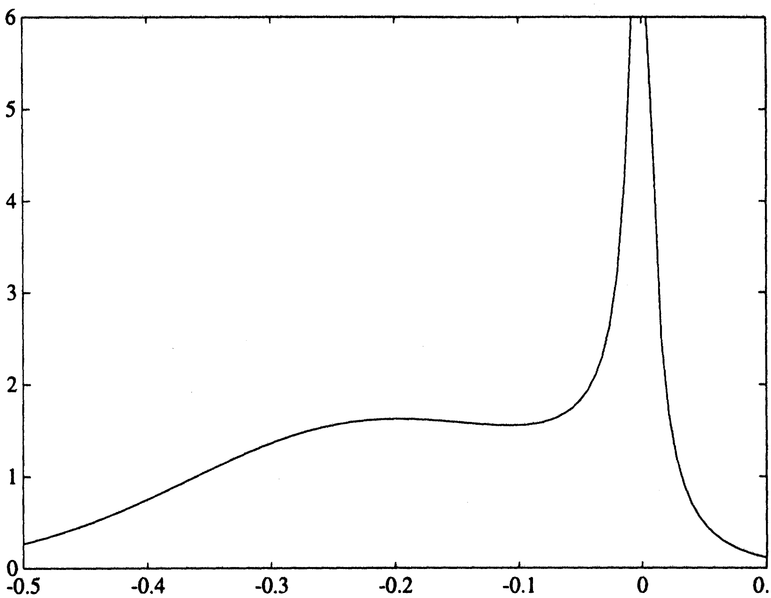

FIGURE 3. Marginal posterior π_{22} demand equation Tintner model.

$$p(\beta|Y,Z) \propto |(y_1 - Y_2\beta)'M_{(Z_1Z_2)}(y_1 - Y_2\beta)|^{(1/2)(T-k_1-k_2-m-1)}$$
$$\times |(y_1 - Y_2\beta)'M_{Z_1}(y_1 - Y_2\beta)|^{-(1/2)(T-k_1-m-1)}, \qquad (2.5)$$

and it has fat tails resulting from the flat nonzero conditional posterior of β given $\Pi_{22} = 0$. For the case of the Tintner model, the marginal posterior is even nonintegrable, which is plausible given the fat tails of the marginal posterior of β shown in Figure 4. In general, the moments of the posterior in (2.5) exist up to/including the degree of overidentification minus 1, implying that exact identified models lead to nonintegrable posteriors when flat (diffuse) priors are used.

A popular method for numerical calculation of posterior densities is to construct the conditional posteriors and use them to perform Gibbs sampling (see Gelfand and Smith, 1989; Smith and Roberts, 1993). When this Markov chain Monte Carlo (MC^2) algorithm is used to compute the marginal posteriors of the parameters of the INSEM, as in Geweke (1996), the local nonidentification problems lead to a reducible Markov chain because when a locally nonidentified parameter value is drawn, the sampler continues drawing nonidentified parameter values. Stated differently, the region of locally nonidentified parameter values is an absorbing state in the Markov chain. The posterior, therefore, violates the convergence conditions for Gibbs samplers as outlined in Roberts and Smith (1994). A solution to this problem is to use informative priors, but this approach is questionable when priors are used that are not in accordance with the likelihood (see Kleibergen, 1997).

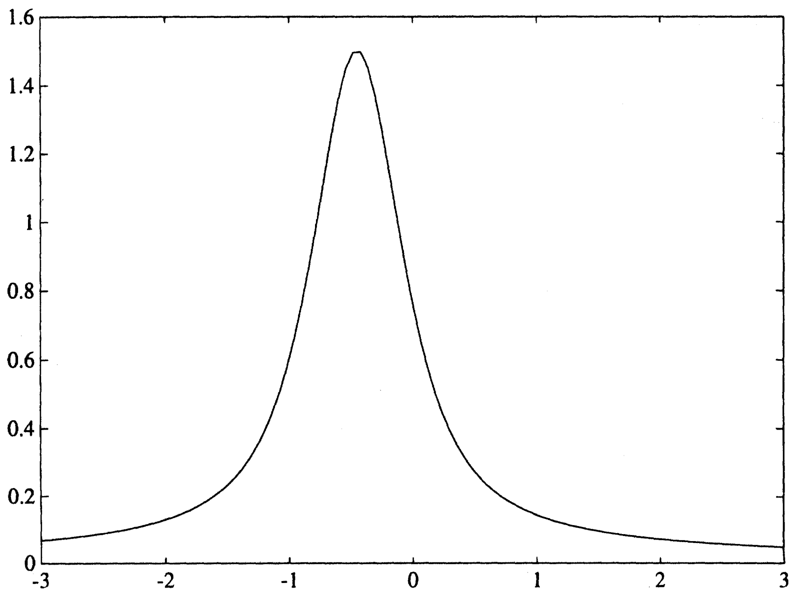

FIGURE 4. Marginal posterior β demand equation Tintner model.

The integrability problems of the posteriors discussed previously result from the dependence of the structural form parameter β on Π_{22}. In classical econometric analysis (see Anderson, 1982; Phillips, 1983; Poirier, 1994), the parameter β is analyzed conditional on a so-called concentration parameter. This is essentially a statistic to test the hypothesis $H_0: \Pi_{22} = 0$, and it shows whether the information in the likelihood is concentrated around $\Pi_{22} = 0$. When this concentration parameter tends to infinity when the sample size becomes large, normal asymptotic theory can be applied (see Anderson, 1982; Phillips, 1983). When $\Pi_{22} = 0$, however, estimators of β, like 2SLS, converge to random variables (see Phillips, 1989). The integrability problems outlined previously show that also in a Bayesian analysis β has to be analyzed given Π_{22}, which is natural given that the identification problems in the likelihood result from model properties, i.e., the nonidentification of β at $\Pi_{22} = 0$, and are not the result of inferior data. Because we know a priori that these integration problems arise, a framework is needed that formalizes the way the parameters are analyzed conditional on one another and that leads to nonpathological posteriors. This framework is constructed in Kleibergen (1997) and is used in the following sections.

3. PRIORS FOR THE INSEM PARAMETERS

In the previous section, we showed that the parameters that suffer from local nonidentification problems should be analyzed conditional on the value of their identifying parameters. This is one of the main properties obtained through the priors constructed in this section. In previous versions of this paper (see Kleibergen and van Dijk, 1992, 1994a; see also Kleibergen and van Dijk, 1994b; Kleibergen and Zivot, 1998; Chao and Phillips, 1998), Jeffreys' priors are used to obtain this property. The resulting posterior when this prior is used is, however, not nested within the assumed posterior of the parameters of the embedding unrestricted linear model. This is a key property of the priors constructed in this section. The prior we construct in this section results from Kleibergen (1997), where it is shown that a whole range of models can be considered as nonlinear restrictions on the parameters of standard linear models. This gives a general framework for the analysis of a large class of models (see also Kleibergen and Hoek, 1996; Kleibergen and Paap, 1997).

3.1. SEM's as Linear Models with Nonlinear Parameter Restrictions

Overidentified SEM's can be considered as a nonlinear restriction on the parameters of a multivariate linear model. It is well known how diffuse and conjugate priors and their resulting posteriors are constructed for the parameters of linear models (see Zellner, 1971). When we explicitly consider the SEM as a nonlinear restriction on the parameters of a linear model, the priors and posteriors of the parameters of the SEM result, straightforwardly, as proportional to the priors and posteriors of the parameters of the linear model under the condition that the restrictions on these parameters hold (see Kleibergen, 1997).

To analyze the restrictions imposed by a SEM on the parameters of a linear model consider the INSEM (2.1) and its implied reduced form (2.2). To show the imposed restrictions, we add a parameter λ to this model that is such that when it is nonzero, (i) there is a one-to-one correspondence with a standard linear model and when it equals zero both (ii) the reduced form of the INSEM results and (iii) it is locally uncorrelated with specific other parameters. This latter property is needed to obtain priors and posteriors of the parameters of the INSEM that are invariant with respect to the specification of the model (see Kleibergen, 1997, for an exact specification of the conditions the restrictions have to satisfy). Several restrictions imposed on the linear model namely lead to the reduced form of the INSEM, but only one restriction leads to priors and posteriors that are invariant with respect to parameter transformations. This invariance property is needed to avoid the Borel–Kolmogorov paradox (see Billingsley, 1986; Drèze and Richard, 1983; for more details on this posterior invariance, see Kleibergen, 1997). The resulting model, which we call the unrestricted SEM, reads

$$(y_1 \quad Y_2) = Z_1(\pi_{11} \quad \Pi_{12}) + Z_2\Pi_{22}(\beta \quad I_{m-1}) + Z_2\Pi_{22\perp}\lambda(\beta \quad I_{m-1})_\perp + (\xi_1 \quad \varepsilon_2), \tag{3.1}$$

where $\lambda:(k_2 - m + 1)\times 1$ and $\Pi_{22\perp}$, $(\beta \; I_{m-1})_\perp$ are the orthogonal complements of Π_{22}, $(\beta \; I_{m-1})$ resp., such that $\Pi_{22}'\Pi_{22\perp} \equiv 0$, $(\beta \; I_{m-1})(\beta \; I_{m-1})_\perp' \equiv 0$, and $\Pi_{22\perp}'\Pi_{22\perp} \equiv I_{k_2-m+1}$, $(\beta \; I_{m-1})_\perp(\beta \; I_{m-1})_\perp' \equiv 1$ (i.e., $\Pi_{22\perp} = (-\Pi_{222}\Pi_{221}^{-1} \; I_{k_2-m+1})'(I_{k_2+m-1} + \Pi_{222}\Pi_{221}^{-1}\Pi_{221}^{-1\prime}\Pi_{222}')^{-1/2}$, when $\Pi_{22} = (\Pi_{221}' \; \Pi_{222}')'$, $\Pi_{221}:(m-1)\times(m-1)$, $\Pi_{222}:(k_2 - m + 1)\times(m-1)$, and $(\beta \; I_{m-1})_\perp = (1 + \beta'\beta)^{-1/2}(1 \; -\beta')$). We note that the orthogonal complements used in other parts of the paper are defined identical to the ones stated previously.

It is clear that when $\lambda = 0$, (3.1) is identical to (2.2) and because λ is multiplied by the orthogonal complements of the matrices containing β and Π_{22}, the information matrix is block diagonal at $\lambda = 0$. We therefore say that λ is locally uncorrelated with β and Π_{22} at $\lambda = 0$. The one-to-one correspondence between the parameters of (3.1) and a multivariate linear model,

$$(y_1 \quad Y_2) = (Z_1 \quad Z_2)\begin{pmatrix} \pi_{11} & \Pi_{12} \\ \phi_1 & \Phi_2 \end{pmatrix} + (\xi_1 \quad \varepsilon_2), \tag{3.2}$$

where $\phi_1 : k_2 \times 1$, $\Phi_2 : k_2 \times (m-1)$, can be shown using a singular value decomposition (SVD) of $\Phi = (\phi_1 \;\; \Phi_2)$ (for definitions of a SVD, see Golub and van Loan, 1989; Magnus and Neudecker, 1988). The equality of (3.2) and (3.1) is shown in Appendix E and uses the SVD of Φ,

$$\Phi = USV', \tag{3.3}$$

where $U : k_2 \times k_2$, $U'U = I_{k_2}$; $V : m \times m$, $V'V = I_m$; and $S : k_2 \times m$ is a rectangular matrix containing the (nonnegative) singular values (in decreasing order) on its main diagonal $(= (s_{11} \ldots s_{mm}))$. If we now write

$$U=\begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix},\qquad S=\begin{pmatrix}S_1 & 0\\ 0 & s_2\end{pmatrix},\quad\text{and}\quad V=\begin{pmatrix}v_{11} & v_{12}\\ V_{21} & v_{22}\end{pmatrix},\tag{3.4}$$

where $U_{11},S_1,V_{21}:(m-1)\times(m-1)$; $v_{12}:1\times 1$; $v_{11}',v_{22}:(m-1)\times 1$; $U_{12}:(m-1)\times(k_2-m+1)$; $U_{21}:(k_2-m+1)\times(m-1)$; $U_{22}:(k_2-m+1)\times(k_2-m+1)$; $s_2:(k_2-m+1)\times 1$, then the following relationship between (Π_{22},β,λ) and (U,S,V) results:

$$\Pi_{22}=\begin{pmatrix}U_{11}\\ U_{21}\end{pmatrix}S_1V_{21}',\qquad \beta=V_{21}'^{-1}v_{11}',\quad\text{and}$$
$$\lambda=(U_{22}U_{22}')^{-1/2}U_{22}s_2v_{12}'(v_{12}v_{12}')^{-1/2}.\tag{3.5}$$

Furthermore, the SVD shows that λ is identified by the smallest singular value of Φ contained in s_2 and is essentially a rotation of s_2 because s_2 is pre- and post-multiplied by orthogonal matrices to obtain λ. Because the singular value s_2 is invariant with respect to the ordering of the variables contained in $Y\,(=(y_1\;\;Y_2))$ and Z_2, the length of λ, which is equal to the length of s_2 because it is a rotation of s_2, is identical for all orderings of the variables contained in Y and Z_2. This property is needed to obtain a prior/posterior of the parameters of the INSEM that is invariant with respect to the ordering of the variables in Y and Z_2.

When we use the least-squares estimator of Φ in (3.3), $\hat{\Phi}=(Z_2'M_{Z_1}Z_2)^{-1}Z_2'M_{Z_1}(y_1\;\;Y_2)$, the estimators for β and Π_{22} resulting from (3.5) are similar to the limited information maximum likelihood estimators (see Anderson and Rubin, 1949; Hausman, 1983) when the instruments are reasonable (for a proof of this, see Kleibergen and Zivot, 1998). The hypothesis $H_0:\lambda=0$ can also be tested in that setting to check the validity of the imposed overidentification.

The preceding discussion shows that the INSEM can be considered as a nonlinear (reduced rank) restriction, $\lambda=0$, on the parameters of the linear model (3.2). We therefore construct the priors and posteriors of the parameters of the INSEM (2.1) as proportional to the priors and posteriors of the parameters of the linear model (3.2) evaluated in $\lambda=0$. This framework for constructing priors and posteriors results from Kleibergen (1997), and we discuss its results for the INSEM in the following (sub)section. The framework can also be used in a full system analysis in which SVD's have to be applied recursively. As this becomes notationally more complicated we discuss it in a later section. Note also that the analysis for exact identified SEM's directly results from the standard linear model because in that case there is a one-to-one correspondence between the parameters of the structural form and the linear model.

3.2. Prior Framework for SEM's

As shown previously, the INSEM can be considered as a nonlinear restriction on the parameters of a multivariate linear model. It is, however, not possible to analytically construct the conditional posterior of the parameters, Ω, π_{11}, β, Π_{12}, and Π_{22}, given the parameter reflecting the restrictions, λ (see Kleibergen, 1997). To show this let $\theta = (\pi_{11}, \beta, \Pi_{12}, \Pi_{22})$ and $\eta = (\Phi, \pi_{11}, \Pi_{12})$, then

$$p_{unsem}(\theta, \lambda | \Omega, Y, Z) \propto p_{lin}(\eta(\theta, \lambda) | \Omega, Y, Z) \left| \frac{\partial \eta}{\partial(\theta, \lambda)} \right|, \tag{3.6}$$

where η is a function of θ and λ and *unsem* stands for unrestricted SEM and *lin* for linear model. Assume that the posterior of η is well behaved, which is typically the case for the posterior of the parameters of a multivariate linear model; then we cannot give an exact expression of the conditional posterior of θ given λ, $p_{unsem}(\theta | \lambda, \Omega, Y, Z)$, including its normalizing constants because we cannot construct the marginal posterior of λ, $p_{unsem}(\lambda | Y, Z)$, analytically. This results as λ is multiplied by $\Pi_{22\perp}$ and $(\beta \;\; I_{m-1})_{\perp}$ in (3.1). The term λ is therefore partly a nonlinear function of β and Π_{22} such that we cannot construct its marginal posterior analytically. So, to obtain a consistent analysis, in the sense that the INSEM has to accord with its embedding linear model, we cannot ignore that the INSEM is a linear model with nonlinear restrictions on its parameters and just proceed by constructing the posterior as in Section 2. In that section we namely implicitly assumed that the involved posterior is proportional to $p_{unsem}(\theta, \lambda | \Omega, Y, Z)|_{\lambda=0}$. This implies a posterior for the parameters of the linear model in $\lambda = 0$,

$$p_{lin}(\eta | \Omega, Y, Z)|_{\lambda=0} \propto p_{unsem}(\eta(\theta, \lambda) | \Omega, Y, Z)|_{\lambda=0} \left| \left(\frac{\partial(\theta, \lambda)}{\partial \eta} \right) \Big|_{\lambda=0} \right|. \tag{3.7}$$

As shown in Section 2 the posterior $p_{unsem}(\theta, \lambda | \Omega, Y, Z)|_{\lambda=0}$ is badly behaved, and the resulting $p_{lin}(\eta | \Omega, Y, Z)|_{\lambda=0}$ is thus also badly behaved. This is, however, a posterior of the parameters of a linear model that is normally well behaved and well understood. It therefore does not belong to (or is nested within) the standard class of posteriors of parameters of linear models. For more details, refer to Kleibergen (1997). Also slight modifications of the INSEM, to, for example, an INSEM that is nested in the original INSEM, lead to a different implied posterior of the parameters of the embedding linear model. We therefore use the priors/posteriors of the parameters of the linear model as a base to construct the priors/posteriors of the parameters of the INSEM. So, we specify a prior for the parameters of the linear model, for example, a diffuse or natural-conjugate prior (see Zellner, 1971), and we evaluate this prior in $\lambda = 0$ to obtain the prior for the INSEM (see Kleibergen, 1997; Kleibergen and Paap, 1997),

$$\begin{aligned} p_{insem}(\theta, \Omega) &\propto p_{unsem}(\theta, \lambda, \Omega)|_{\lambda=0} \\ &\propto p_{lin}(\eta(\theta, \lambda), \Omega)|_{\lambda=0} \left| \left(\frac{\partial \eta}{\partial(\theta, \lambda)} \right) \Big|_{\lambda=0} \right|, \end{aligned} \tag{3.8}$$

where *insem* stands for INSEM. We note that we can also perform the construction of the prior vice versa by constructing a prior on the structural form parameters and checking whether the implied prior on the parameters of the embedding linear model is plausible (see Kleibergen, 1997; Kleibergen and Zivot, 1998).

3.2.1. Diffuse prior. Using the framework resulting from (3.8), a diffuse (Jeffreys') prior for the parameters of the linear model, $(\pi_{11}, \Pi_{12}, \Phi, \Omega)$,

$$p_{lin}(\pi_{11}, \Pi_{12}, \Phi, \Omega) \propto |\Omega|^{-(1/2)(k+m+1)} \propto |\Omega|^{-(1/2)(m+1)} |\Omega^{-1} \otimes Z'Z|^{1/2}, \tag{3.9}$$

implies the prior for the parameters of the INSEM, $(\beta, \pi_{11}, \Pi_{12}, \Pi_{22}, \Omega)$,

$$\begin{aligned} &p_{insem}(\beta, \pi_{11}, \Pi_{12}, \Pi_{22}, \Omega) \\ &\quad \propto |\Omega|^{-(1/2)(m+1)} |\Omega^{-1} \otimes Z'Z|^{1/2} |J(\Phi, (\Pi_{22}, \beta, \lambda))|_{\lambda=0}| \\ &\quad \propto |\Omega|^{-(1/2)(k_1+m+1)} |Z_1'Z_1|^{(1/2)m} |B_\perp \Omega B_\perp'|^{-(1/2)(k_2-m+1)} \\ &\quad \times |\Pi_{22\perp}'(Z_2'M_{Z_1}Z_2)^{-1}\Pi_{22\perp}|^{-1/2} \\ &\quad \times \left| \begin{pmatrix} B\Omega^{-1}B' \otimes Z_2'M_{Z_1}Z_2 & B\Omega^{-1}e_1 \otimes Z_2'M_{Z_1}Z_2\Pi_{22} \\ e_1'\Omega^{-1}B' \otimes \Pi_{22}'Z_2'M_{Z_1}Z_2 & e_1'\Omega^{-1}e_1 \otimes \Pi_{22}'Z_2'M_{Z_1}Z_2\Pi_{22} \end{pmatrix} \right|^{1/2}, \end{aligned} \tag{3.10}$$

where $e_1 : m \times 1$ is the first m-dimensional unity vector, $B = (\beta \; I_{m-1})$, $|J(\Phi, (\Pi_{22}, \beta, \lambda))| = |\partial\eta / \partial(\theta, \lambda)|$ and is constructed in Appendix A.

The prior (3.10) shows that β is analyzed conditional on the value of Π_{22}, as it should be according to the local nonidentification of β for lower-rank values of Π_{22}. Furthermore, the prior shows the functional form of a diffuse prior for the parameters of the INSEM. This accords with our conclusions from the previous section that diffuseness for models like the INSEM has to be defined in a different way than the usual one for parameters of linear models.

We note that the prior (3.10) is the Jeffreys' prior of the unrestricted reduced form of the INSEM (3.1) evaluated in $\lambda = 0$. In Kleibergen and van Dijk (1994a), the Jeffreys' prior of the reduced form of the INSEM (2.2) is used to obtain well-behaved posteriors (see also Chao and Phillips, 1998). This prior is apart from $|B_\perp \Omega B_\perp'|^{-(1/2)(k_2-m+1)} |\Pi_{22\perp}'(Z_2'M_{Z_1}Z_2)^{-1}\Pi_{22\perp}|^{-1/2}$ identical to (3.10). We use (3.10) instead of that prior for three reasons. First, (3.10) results in a generic manner from the linear model (3.2). Second, the concept for constructing (3.10) can also be applied in the full system analysis whereas the Jeffreys' priors of the reduced forms of full system SEM's are intractable. Third, although we use data matrices in (3.10) to obtain a more interpretable expression of the prior, it is not data dependent as no data matrices appear in the Jacobian $J(\Phi, (\Pi_{22}, \beta, \lambda))$, and $|Z'Z|$ can just be left out. A Jeffreys' prior on the reduced form (2.2) is data dependent however. For more details, see Kleibergen and Zivot (1998).

The prior (3.10) is identical to the Jeffreys' prior for the reduced form of the orthonormal SEM (see Phillips, 1983; Chao and Phillips, 1996), where $\Omega = I_m$ and $Z_2'M_{Z_1}Z_2 = I_{k_2}$, as $B_\perp B_\perp' = 1$, $\Pi_{22\perp}'\Pi_{22\perp} = I_{k_2-m+1}$. Using Rayleigh quotients it can also be shown that the ratio of the prior (3.10) and a Jeffreys' prior on the reduced form (2.2) is bounded between finite nonzero constants.

3.2.2. Natural conjugate prior. In case of a natural conjugate prior for the parameters of the linear model, we specify an inverted-Wishart prior for Ω and a matrix normal prior for (π_{11},Π_{12},Φ) given Ω,

$$p_{lin}(\Omega) \propto |G|^{(1/2)h}|\Omega|^{-(1/2)(h+m+1)}\exp[-\tfrac{1}{2}\mathrm{tr}(\Omega^{-1}G)]$$

$$\begin{aligned} p_{lin}(\pi_{11},\Pi_{12},\Phi|\Omega) \propto |\Omega|^{-(1/2)m}|A|^{(1/2)k} \\ \times \exp\left[-\frac{1}{2}\mathrm{tr}\left(\Omega^{-1}\left(\begin{pmatrix}\pi_{11} & \Pi_{12}\\ \varphi_1 & \Phi_2\end{pmatrix}-P\right)'\right.\right. \\ \left.\left.\times A\left(\begin{pmatrix}\pi_{11} & \Pi_{12}\\ \varphi_1 & \Phi_2\end{pmatrix}-P\right)\right)\right], \end{aligned} \quad (3.11)$$

where $G: m\times m$, $A: k\times k$, G and A are positive definite symmetric (pds) matrices, $P: k\times m$, and h is the prior degrees of freedom parameter. The matrix A can be decomposed as

$$A=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}, \quad (3.12)$$

where $A_{11}: k_1\times k_1, A_{12}=A_{21}': k_2\times k_1, A_{22}: k_2\times k_2$. The prior of the parameters of the INSEM resulting from $p_{lin}(\pi_{11},\Pi_{12},\Phi,\Omega)$ can again be constructed using (3.8),

$$\begin{aligned} &p_{insem}(\beta,\pi_{11},\Pi_{12},\Pi_{22},\Omega) \\ &\quad\propto p_{lin}(\pi_{11},\Pi_{12},\Phi(\beta,\Pi_{22},\lambda),\Omega|_{\lambda=0}|J(\Phi,(\Pi_{22},\beta,\lambda))|_{\lambda=0}| \\ &\quad\propto |G|^{(1/2)h}|\Omega|^{-(1/2)(h+k_1+m+1)}|A_{11}|^{(1/2)m}|B_\perp\Omega B_\perp'|^{-(1/2)(k_2-m+1)} \\ &\quad\times|\Pi_{22\perp}'A_{22.1}^{-1}\Pi_{22\perp}|^{-1/2} \\ &\quad\times\left|\begin{pmatrix}B\Omega^{-1}B'\otimes A_{22.1} & B\Omega^{-1}e_1\otimes A_{22.1}\Pi_{22}\\ e_1'\Omega^{-1}B'\otimes\Pi_{22}'A_{22.1} & e_1'\Omega^{-1}e_1\otimes\Pi_{22}'A_{22.1}\Pi_{22}\end{pmatrix}\right|^{1/2} \\ &\quad\times\exp\left[-\frac{1}{2}\mathrm{tr}\left(\Omega^{-1}\left(G+\left(\begin{pmatrix}\pi_{11} & \Pi_{12}\\ \Pi_{22}\beta & \Pi_{22}\end{pmatrix}-P\right)'\right.\right.\right. \\ &\quad\left.\left.\left.\times A\left(\begin{pmatrix}\pi_{11} & \Pi_{12}\\ \Pi_{22}\beta & \Pi_{22}\end{pmatrix}-P\right)\right)\right)\right], \end{aligned} \quad (3.13)$$

where $A_{22.1}=A_{22}-A_{21}A_{11}^{-1}A_{12}$ and the specification of (3.13) is not unique in the sense that certain scaling factors are used to obtain a more interpretable expression.

It may be that we have more knowledge about possible values of the parameters of the INSEM than about the parameters of the linear model. This knowledge can be used in the construction of the prior of the parameters of the linear model though, as these parameters are an exact function of the parameters of the INSEM when the restriction $\lambda=0$ holds. We can also directly specify a prior on the parameters of the INSEM and check whether the implied prior on the parameters of the embedding linear model is plausible (see Kleibergen and Zivot, 1998).

The prior (3.13) does not belong to a known class of probability density functions, and we do not know analytical expressions of its moments (which only exist up to its first order (but not including it)) or normalizing constant. These properties can be calculated using Monte Carlo simulation, and in Section 5 we construct a simulation algorithm to obtain drawings from (3.13).

4. POSTERIORS OF THE INSEM PARAMETERS

The framework for constructing the priors of the parameters of the INSEM can directly be applied to construct the posteriors of the parameters of the INSEM. This results because the likelihood of the INSEM is a continuous function of the parameters such that the posterior, which is proportional to the product of the prior and the likelihood, can be evaluated in the same way as the prior,

$$\begin{aligned}
&p_{insem}(\beta,\pi_{11},\Pi_{12},\Pi_{22},\Omega|Y,Z)\\
&\quad\propto p_{insem}(\beta,\pi_{11},\Pi_{12},\Pi_{22},\Omega)L_{insem}(Y|\beta,\pi_{11},\Pi_{12},\Pi_{22},\Omega,Z)\\
&\quad\propto p_{unsem}(\beta,\lambda,\pi_{11},\Pi_{12},\Pi_{22},\Omega)|_{\lambda=0}L_{unsem}(Y|\beta,\lambda,\pi_{11},\Pi_{12},\Pi_{22},\Omega,Z)|_{\lambda=0}\\
&\quad\propto p_{lin}(\pi_{11},\Pi_{12},\Phi(\beta,\lambda,\Pi_{22}),\Omega)|_{\lambda=0}|J(\Phi,(\Pi_{22},\beta,\lambda))|_{\lambda=0}|\\
&\qquad\times L_{lin}(Y|\pi_{11},\Pi_{12},\Phi(\beta,\lambda,\Pi_{22}),\Omega,Z)|_{\lambda=0}.
\end{aligned}\tag{4.1}$$

In the following two subsections, we construct the posteriors for different specifications of the prior, i.e., a diffuse and natural conjugate prior.

4.1. Posterior INSEM Using Diffuse Prior

Using the diffuse prior (3.10), the joint posterior of the parameters of the INSEM can directly be constructed from this prior and the likelihood using (4.1),

$$\begin{aligned}
&p_{insem}(\beta,\pi_{11},\Pi_{12},\Pi_{22},\Omega|Y,Z)\\
&\quad\propto p_{insem}(\beta,\pi_{11},\Pi_{12},\Pi_{22},\Omega)L(Y|\beta,\pi_{11},\Pi_{12},\Pi_{22},\Omega,Z)\\
&\quad\propto|\Omega|^{-(1/2)(T+k_1+m+1)}|Z_1'Z_1|^{(1/2)m}|B_\perp\Omega B_\perp'|^{-(1/2)(k_2-m+1)}\\
&\qquad\times|\Pi_{22\perp}'(Z_2'M_{Z_1}Z_2)^{-1}\Pi_{22\perp}|^{-1/2}\\
&\qquad\times\left|\begin{pmatrix}B\Omega^{-1}B'\otimes Z_2'M_{Z_1}Z_2 & B\Omega^{-1}e_1\otimes Z_2'M_{Z_1}Z_2\Pi_{22}\\ e_1'\Omega^{-1}B'\otimes\Pi_{22}'Z_2'M_{Z_1}Z_2 & e_1'\Omega^{-1}e_1\otimes\Pi_{22}'Z_2'M_{Z_1}Z_2\Pi_{22}\end{pmatrix}\right|^{1/2}\\
&\qquad\times\exp\left[-\frac{1}{2}\operatorname{tr}\left(\Omega^{-1}\left((y_1\quad Y_2)-(Z_1\quad Z_2)\begin{pmatrix}\pi_{11} & \Pi_{12}\\ \Pi_{22}\beta & \Pi_{22}\end{pmatrix}\right)'\right.\right.\\
&\qquad\qquad\left.\left.\times\left((y_1\quad Y_2)-(Z_1\quad Z_2)\begin{pmatrix}\pi_{11} & \Pi_{12}\\ \Pi_{22}\beta & \Pi_{22}\end{pmatrix}\right)\right)\right].
\end{aligned}\tag{4.2}$$

The posterior (4.2) does not belong to a known class of probability density functions nor do any of the conditional posteriors, apart from the conditional posterior of (π_{11},Π_{12}) given (β,Π_{22},Ω), which is matrix normal, belong to a known class of probability density functions. So, we can only analytically integrate out (π_{11},Π_{12}) to obtain the marginal posterior of (β,Π_{22},Ω),

$$\begin{aligned} p_{insem}(\beta,\Pi_{22},\Omega|Y,Z) &\propto |\Omega|^{-(1/2)(T+m+1)}|B_\perp\Omega B_\perp'|^{-(1/2)(k_2-m+1)} \\ &\times \left|\begin{pmatrix} B\Omega^{-1}B'\otimes Z_2'M_{Z_1}Z_2 & B\Omega^{-1}e_1\otimes Z_2'M_{Z_1}Z_2\Pi_{22} \\ e_1'\Omega^{-1}B'\otimes \Pi_{22}'Z_2'M_{Z_1}Z_2 & e_1'\Omega^{-1}e_1\otimes \Pi_{22}'Z_2'M_{Z_1}Z_2\Pi_{22}\end{pmatrix}\right|^{1/2} \\ &\times |\Pi_{22\perp}'(Z_2'M_{Z_1}Z_2)^{-1}\Pi_{22\perp}|^{-1/2} \\ &\times \exp[-\tfrac{1}{2}\mathrm{tr}(\Omega^{-1}((y_1\quad Y_2)-Z_2\Pi_{22}(\beta\quad I_{m-1}))' \\ &\qquad \times M_{Z_1}((y_1\quad Y_2)-Z_2\Pi_{22}(\beta\quad I_{m-1})))], \end{aligned} \tag{4.3}$$

which shows the functional form of the kernel of the density of a matrix normal distributed random matrix with reduced rank (see Kleibergen, 1997). The posterior (4.3) is proportional to the product of the marginal posterior of (Φ,Ω) and the Jacobian of the transformation evaluated in $\lambda=0$,

$$p_{insem}(\beta,\Pi_{22},\Omega|Y,Z) \propto p_{lin}(\Phi(\beta,\lambda,\Pi_{22}),\Omega|Y,Z)|_{\lambda=0}|J(\Phi,(\Pi_{22},\beta,\lambda))|_{\lambda=0}|. \tag{4.4}$$

In Section 5, we construct importance and Metropolis–Hastings samplers for calculating the marginal posteriors of (4.3) that use (4.4).

4.2. Posterior INSEM Using Natural Conjugate Prior

Identically to the posterior of the parameters of the INSEM using a diffuse prior (4.2), we can construct the posterior of the parameters of the INSEM when we use the natural conjugate prior (3.13),

$$\begin{aligned} &p_{insem}(\beta,\pi_{11},\Pi_{12},\Pi_{22},\Omega|Y,Z) \\ &\propto |\Omega|^{-(1/2)(T+k_1+m+1)}|(A+Z'Z)_{11}|^{(1/2)m}|B_\perp\Omega B_\perp'|^{-(1/2)(k_2-m+1)} \\ &\times \left|\begin{pmatrix} B\Omega^{-1}B'\otimes (A+Z'Z)_{22.1} & B\Omega^{-1}e_1\otimes (A+Z'Z)_{22.1}\Pi_{22} \\ e_1'\Omega^{-1}B'\otimes \Pi_{22}'(A+Z'Z)_{22.1} & e_1'\Omega^{-1}e_1\otimes \Pi_{22}'(A+Z'Z)_{22.1}\Pi_{22}\end{pmatrix}\right|^{1/2} \\ &\times |\Pi_{22\perp}'(A+Z'Z)_{22.1}^{-1}\Pi_{22\perp}|^{-1/2} \\ &\times \exp\left[-\frac{1}{2}\mathrm{tr}\left(\Omega^{-1}\left(\tilde{G}+\left(\begin{pmatrix}\pi_{11} & \Pi_{12}\\ \Pi_{22}\beta & \Pi_{22}\end{pmatrix}-\tilde{\Pi}\right)'\right.\right.\right. \\ &\qquad\qquad \left.\left.\left.\times (A+Z'Z)\left(\begin{pmatrix}\pi_{11} & \Pi_{12}\\ \Pi_{22}\beta & \Pi_{22}\end{pmatrix}-\tilde{\Pi}\right)\right)\right)\right], \end{aligned} \tag{4.5}$$

where $\tilde{\Pi}=(A+Z'Z)^{-1}(Z'Y+A'P)$, $\tilde{G}=G+Y'Y-\tilde{\Pi}'(A+Z'Z)\tilde{\Pi}$, $Y=(y_1\quad Y_2)$. Again similar to the posterior using a diffuse prior (4.2), only the conditional posterior of (π_{11}, Π_{12}) given $(\beta, \Pi_{22}, \Omega)$ belongs to a known class of probability density functions, and (π_{11}, Π_{12}) are the only parameters that can be integrated out analytically to obtain the marginal posterior of $(\beta, \Pi_{22}, \Omega)$,

$$\begin{aligned} p_{insem}(\beta, \Pi_{22}, \Omega | Y, Z) &\propto |\Omega|^{-(1/2)(T+m+1)} |B_{\perp}\Omega B_{\perp}'|^{-(1/2)(k_2-m+1)} \\ &\times \left| \begin{pmatrix} B\Omega^{-1}B' \otimes (A+Z'Z)_{22.1} & B\Omega^{-1}e_1 \otimes (A+Z'Z)_{22.1}\Pi_{22} \\ e_1'\Omega^{-1}B' \otimes \Pi_{22}'(A+Z'Z)_{22.1} & e_1'\Omega^{-1}e_1 \otimes \Pi_{22}'(A+Z'Z)_{22.1}\Pi_{22} \end{pmatrix} \right|^{1/2} \\ &\times |\Pi_{22\perp}'(A+Z'Z)_{22.1}^{-1}\Pi_{22\perp}|^{-1/2} \\ &\times \exp[-\tfrac{1}{2}\mathrm{tr}(\Omega^{-1}(\tilde{G} + (\Pi_{22}(\beta \quad I_{m-1}) - \tilde{\Pi}_2)' \\ &\qquad \times (A+Z'Z)_{22.1}(\Pi_{22}(\beta \quad I_{m-1}) - \tilde{\Pi}_2)))], \end{aligned} \tag{4.6}$$

where $\tilde{\Pi} = (\tilde{\Pi}_1' \;\; \tilde{\Pi}_2')'$, $\tilde{\Pi}_1 : k_1 \times m$, $\tilde{\Pi}_2 : k_2 \times m$.

Again (4.4) applies to this posterior, and we use it in the following section to construct a posterior simulator.

5. SIMULATING POSTERIORS

As mentioned before the posteriors (4.3) and (4.6) do not belong to a standard class of probability density functions nor do their conditional posteriors. We can therefore not perform Gibbs sampling as the conditional posteriors are nonstandard. The simulation algorithms constructed in this section therefore generate drawings from a probability density function that approximates the true posterior. To correct for not drawing from the true posterior, weights are attached to each drawing of the parameters proportional to the ratio of the posterior and the approximating density in the generated parameter points. These weights can be used in both importance sampling algorithms (see Kloek and van Dijk, 1978; Geweke, 1989) and Metropolis–Hastings algorithms (see Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller, 1953; Hastings, 1970) to draw from the posterior. We first discuss the construction of the weights and the approximating density, and hereafter we briefly discuss the two different simulation algorithms.

We use the posterior of the unrestricted SEM, $p_{unsem}(\beta, \lambda, \Pi_{22}, \Omega | Y, Z)$, as approximating density of the posterior of the INSEM, $p_{insem}(\beta, \Pi_{22}, \Omega | Y, Z)$. The posterior of the unrestricted SEM contains the parameter λ, however, which is not present in the posterior of the INSEM. To obtain a density that both accords with the posterior of the INSEM and contains λ, we assume that λ is generated given $(\beta, \Pi_{22}, \Omega)$ from a proper conditional density $g(\lambda | \beta, \Pi_{22}, \Omega)$, which we specify ourselves (see Chen, 1994; Verdinelli and Wasserman, 1995; Kleibergen, 1997; Kleibergen and Paap, 1997). Furthermore, we assume that β, Π_{22}, and Ω are generated from $p_{insem}(\beta, \Pi_{22}, \Omega | Y, Z)$. So, as density function to be approximated by $p_{unsem}(\beta, \lambda, \Pi_{22}, \Omega | Y, Z)$ we have

$$\begin{aligned} &g(\lambda | \beta, \Pi_{22}, \Omega) p_{insem}(\beta, \Pi_{22}, \Omega | Y, Z) \\ &\quad \propto g(\lambda | \beta, \Pi_{22}, \Omega)(p_{unsem}(\beta, \lambda, \Pi_{22}, \Omega | Y, Z)|_{\lambda=0}). \end{aligned} \tag{5.1}$$

The weight function thus becomes

$$w(\beta,\lambda,\Pi_{22},\Omega) = \frac{g(\lambda|\beta,\Pi_{22},\Omega)(p_{unsem}(\beta,\lambda,\Pi_{22},\Omega|Y,Z)|_{\lambda=0})}{p_{unsem}(\beta,\lambda,\Pi_{22},\Omega|Y,Z)}. \tag{5.2}$$

The quality of the approximating density $p_{unsem}(\beta,\lambda,\Pi_{22},\Omega|Y,Z)$ crucially depends on the chosen specification of $g(\lambda|\beta,\Pi_{22},\Omega)$. In case we use the diffuse prior for the parameters of the INSEM (3.10), a natural choice of $g(\lambda|\beta,\Pi_{22},\Omega)$ is

$$\begin{aligned} g(\lambda|\beta,\Pi_{22},\Omega) &= (2\pi)^{-(1/2)(k_2-m+1)}|B_\perp\Omega^{-1}B_\perp'|^{(1/2)(k_2-m+1)}|\Pi_{22\perp}'Z_2'M_{Z_1}Z_2\Pi_{22\perp}|^{1/2} \\ &\quad\times \exp[-\tfrac{1}{2}\mathrm{tr}(B_\perp\Omega^{-1}B_\perp'(\lambda-\hat\lambda)'\Pi_{22\perp}'Z_2'M_{Z_1}Z_2\Pi_{22\perp}(\lambda-\hat\lambda))], \end{aligned} \tag{5.3}$$

where

$$\hat\lambda = (\Pi_{22\perp}'Z_2'M_{Z_1}Z_2\Pi_{22\perp})^{-1}\Pi_{22\perp}'Z_2'M_{Z_1}(Y-Z_2\Pi_{22}B)\Omega^{-1}B_\perp'(B_\perp\Omega^{-1}B_\perp')^{-1},$$

whereas

$$\begin{aligned} g(\lambda|\beta,\Pi_{22},\Omega) &= (2\pi)^{-(1/2)(k_2-m+1)}|B_\perp\Omega^{-1}B_\perp'|^{(1/2)(k_2-m+1)}|\Pi_{22\perp}'(A+Z'Z)_{22.1}\Pi_{22\perp}|^{1/2} \\ &\quad\times \exp[-\tfrac{1}{2}\mathrm{tr}(B_\perp\Omega^{-1}B_\perp'(\lambda-\hat\lambda)'\Pi_{22\perp}'(A+Z'Z)_{22.1}\Pi_{22\perp}(\lambda-\hat\lambda))], \end{aligned} \tag{5.4}$$

where $\hat\lambda = (\Pi_{22\perp}'(A+Z'Z)_{22.1}\Pi_{22\perp})^{-1}\Pi_{22\perp}'((AP+Z'Y)_2-(A+Z'Z)_{22.1}\Pi_{22}B)\times \Omega^{-1}B_\perp'(B_\perp\Omega^{-1}B_\perp')^{-1}$, $AP + Z'Y = ((AP + Z'Y)_1' \; (AP + Z'Y)_2')'$, $(AP + Z'Y)_1 : k_1\times m$, $(AP+Z'Y)_2 : k_2\times m$, is a natural choice of $g(\lambda|\beta,\Pi_{22},\Omega)$ when we use the natural conjugate prior (3.11).

The weight function resulting from these choices of g reads in both cases

$$w(\beta,\lambda,\Pi_{22},\Omega) = \frac{|J(\Phi,(\beta,\lambda,\Pi_{22}))|_{\lambda=0}|}{|J(\Phi,(\beta,\lambda,\Pi_{22}))|}\, g(\lambda|\beta,\Pi_{22},\Omega)|_{\lambda=0}, \tag{5.5}$$

where $g(\lambda|\beta,\Pi_{22},\Omega)$ should be chosen from (5.3) and (5.4) according to the involved prior. In Appendix A, we show that $|J(\Phi,(\beta,\lambda,\Pi_{22}))| \geq |J(\Phi,(\beta,\lambda,\Pi_{22}))|_{\lambda=0}|$ such that the ratio of the Jacobians in (5.5) is always finite. Furthermore as $g(\lambda|\beta,\Pi_{22},\Omega)|$ is a proper conditional density, it is also finite, and the weight function is consequently always finite.

When $\lambda = 0$, the ratio of Jacobians in (5.5) is equal to one, and the weight function then simplifies to the proposed conditional density of λ evaluated in $\lambda = 0$. The weight function is therefore always finite and nonzero when $\lambda = 0$. All drawings of (β,λ,Π_{22}) for which $\lambda = 0$ thus get a finite nonzero weight. This has consequences for the existence of moments of the posterior $p_{insem}(\beta,\Pi_{22},\Omega|Y,Z)$ because it implies that the degree of finite moments is determined by the transformation of Φ to (β,λ,Π_{22}). According to (3.5), $\beta = V_{21}'^{-1}v_{11}'$. As no restrictions are imposed on the rank of V_{21}, this implies that the posterior of β has Cauchy-type tails and no finite mean and variance (see Kleibergen and Zivot, 1998). Note also that the sampling distributions of limited information maximum likelihood estimators have Cauchy-type tails (see Anderson, 1982; Phillips, 1983).

We summarize the different steps involved in obtaining the weight function, attached to the ith drawing, $i = 1,\dots,N$, in a simulation algorithm as follows (see also Kleibergen, 1997; Kleibergen and Paap, 1997),

Draw Ω^i from $p_{lin}(\Omega|Y,Z)$.
Draw Φ^i from $p_{lin}(\Phi|\Omega^i,Y,Z)$.
Perform a singular value decomposition of $\Phi^i = U^i S^i V^{i\prime}$.
Compute β, λ, Π_{22} according to (3.4) and (3.5).
Compute $w(\beta^i,\lambda^i,\Pi_{22}^i,\Omega^i)$ according to (5.5).
Draw π_{11}^i, Π_{12}^i from $p_{lin}(\pi_{11},\Pi_{12}|\Phi(\beta^i,\lambda,\Pi_{22}^i),\Omega^i,Y,Z)|_{\lambda=0}$.

The posteriors of the linear model parameters, Ω and Φ, used in the first step are standard density functions, i.e., inverted-Wishart and matrix normal, respectively, in case of diffuse or natural conjugate priors. The exact functional specification of these densities depends on the specification of the involved priors and is straightforward to construct, i.e.,

$$p_{lin}(\Omega|Y,Z) \propto |\Omega|^{-(1/2)(T+l+m+1)} \exp[-\tfrac{1}{2}\mathrm{tr}(\Omega^{-1}Q)], \qquad (5.6)$$

where $l = 0$, $Q = Y'M_Z Y$ in the case of the diffuse prior, $l = h$ and $Q = \tilde{G}$ in the case of the natural conjugate prior, and

$$p_{lin}(\Phi|\Omega,Y,Z) \propto |\Omega|^{-(1/2)k_2} \exp[-\tfrac{1}{2}\mathrm{tr}(\Omega^{-1}(\Phi-\hat{\Phi})'W(\Phi-\hat{\Phi})], \qquad (5.7)$$

where $\hat{\Phi} = (Z_2'M_{Z_1}Z_2)^{-1}Z_2'M_{Z_1}Y$, $W = Z_2'M_{Z_1}Z_2$ in the case of the diffuse prior and $\hat{\Phi} = \tilde{\Pi}_2$, $W = (A + Z'Z)_{22.1}$ in the case of the natural conjugate prior. In Kleibergen and Zivot (1998) another simulation algorithm to generate drawings from the posterior $p_{insem}(\beta,\Pi_{22},\Omega|Y,Z)$ is constructed that is sometimes more efficient but is more difficult to generalize to the full information case.

The weight function can either be used in an importance or a Metropolis–Hastings sampling algorithm to calculate the marginal posteriors or moments of these. Using the importance sampling algorithm (see Kloek and van Dijk, 1978; Geweke, 1989), we approximate the moment $E(f(\pi_{11},\Pi_{12},\beta,\Pi_{22},\Omega))$ by

$$\hat{E}(f(\pi_{11},\Pi_{12},\beta,\Pi_{22},\Omega)) = \frac{\sum_{i=1}^{N} w(\beta^i,\lambda^i,\Pi_{22}^i,\Omega^i)f(\pi_{11}^i,\Pi_{12}^i,\beta^i,\Pi_{22}^i,\Omega^i)}{\sum_{i=1}^{N} w(\beta^i,\lambda^i,\Pi_{22}^i,\Omega^i)}, \qquad (5.8)$$

where we use $\hat{E}$ to indicate that it is an estimator of the true expectation E. In Geweke (1989), it is shown that under quite general conditions central limit theorems can be used to prove the convergence of the approximation (5.8) to its true value. As the weights are always finite, they satisfy the conditions for the central limit theorems to apply, and statistics can be calculated that show the numerical accuracy of the approximation (5.8).

The weights (5.2) can also be used in a Metropolis–Hastings (M-H) algorithm (see Metropolis et al., 1953; Hastings, 1970) known as the independence sampler (see Tierney, 1994). This algorithm constructs a Markov chain from the drawn $(\pi_{11}^i,\Pi_{12}^i,\beta^i,\Pi_{22}^i,\Omega^i)$'s. The $(\pi_{11}^i,\Pi_{12}^i,\beta^i,\Pi_{22}^i,\Omega^i)$'s in this Markov chain are accepted as drawings from the posterior. This is achieved using the following steps:

0. $i=1$.
1. Draw $(\pi_{11}^{i+1},\Pi_{12}^{i+1},\beta^{i+1},\Pi_{22}^{i+1},\Omega^{i+1})$ using the simulation scheme stated previously. Given that $(\pi_{11}^i,\Pi_{12}^i,\beta^i,\Pi_{22}^i,\Omega^i)$ is accepted as drawing from the posterior, $(\pi_{11}^{i+1},\Pi_{12}^{i+1},\beta^{i+1},\Pi_{22}^{i+1},\Omega^{i+1})$ is accepted as the $(i+1)$th drawing from the posterior with probability $\min((w(\beta^i,\lambda^i,\Pi_{22}^i,\Omega^i))/(w(\beta^{i+1},\lambda^{i+1},\Pi_{22}^{i+1},\Omega^{i+1})),1)$; otherwise $(\pi_{11}^{i+1},\Pi_{12}^{i+1},\beta^{i+1},\Pi_{22}^{i+1},\Omega^{i+1}) = (\pi_{11}^i,\Pi_{12}^i,\beta^i,\Pi_{22}^i,\Omega^i)$.
2. $i=i+1$. Go to 1.

When the resulting Markov chain, $(\pi_{11}^i,\Pi_{12}^i,\beta^i,\Pi_{22}^i,\Omega^i)$, $i=1,\ldots$ has converged to its equilibrium distribution, say after H drawings, we can record $(\pi_{11}^i,\Pi_{12}^i,\beta^i,\Pi_{22}^i,\Omega^i)$, $i=H+1,\ldots$ as simulated values of the parameters from the posterior.

The simulation algorithms can also be used to calculate other properties of the posterior, like Bayes factors and Bayesian Lagrange multiplier statistics (see Kleibergen and Paap, 1997), and to obtain drawings from the natural conjugate prior (3.13). In that case, the natural choice of the involved $g(\lambda\,|\,\beta,\Pi_{22},\Omega)$ reads

$$g(\lambda\,|\,\beta,\Pi_{22},\Omega) = (2\pi)^{-(1/2)(k_2-m+1)}\,|B_\perp\Omega^{-1}B_\perp'|^{(1/2)(k_2-m+1)}\,|\Pi_{22\perp}'A_{22.1}\Pi_{22\perp}|^{1/2} \times \exp[-\tfrac{1}{2}\mathrm{tr}(B_\perp\Omega^{-1}B_\perp'(\lambda-\hat{\lambda})'\Pi_{22\perp}'A_{22.1}\Pi_{22\perp}(\lambda-\hat{\lambda}))], \tag{5.9}$$

where $\hat{\lambda} = (\Pi_{22\perp}'A_{22.1}\Pi_{22\perp})^{-1}\Pi_{22\perp}'A_{22.1}(P_2-\Pi_{22}B)\Omega^{-1}B_\perp'(B_\perp\Omega^{-1}B_\perp')^{-1}$, $P = (P_1' \;\; P_2')'$, $P_1\!: k_1\times m$, $P_2\!: k_2\times m$, and $p_{lin}(\Omega\,|\,Y,Z)$, $p_{lin}(\Phi\,|\,\Omega,Y,Z)$ both result from (3.11). This also shows the conjugateness of this prior as it equals the posterior using a diffuse prior of some arbitrary set of observations that does not hold for the extended natural conjugate priors, which are also specified for SEM's, used by Drèze and Morales (1976) and Drèze and Richard (1983). We note that the simulation algorithms do not calculate γ; as $\gamma=\pi_{11}+\Pi_{12}\beta$, we can easily incorporate γ into these algorithms.

6. FULL SYSTEM ANALYSIS

The INSEM is a reduced rank restriction on a parameter matrix of a linear model. A full system analysis of a SEM can also be specified as a linear model with nonlinear restrictions on its parameters. Again these restrictions are reduced rank restrictions, but the difference with the INSEM is that they can depend on one another in a recursive way. Theorem 1 states that the reduced form of a SEM is a linear model with reduced rank restrictions on its parameter matrices.

THEOREM 6.1. *Assume that a* SEM *has the following specification:*

$$(Y_{\bar{m}} \quad Y_m)\begin{pmatrix} B_{\bar{m}\bar{m}} & B_{\bar{m}m} \\ B_{m\bar{m}} & B_{mm} \end{pmatrix} = (Z_{\bar{m}} \quad Z_{\bar{m}m} \quad Z_m)\begin{pmatrix} \Gamma_{\bar{m}\bar{m}} & 0 \\ \Gamma_{m\bar{m}} & \Gamma_{\bar{m}m} \\ 0 & \Gamma_{mm} \end{pmatrix} + (\varepsilon_{\bar{m}} \quad \varepsilon_m), \tag{6.1}$$

where the number of variables contained in Y_m *is chosen such that* $\Gamma_{mm}: i_m \times j_m$ $(i_m \geq j_m)$ *and* $\Gamma_{\bar{m}m}: l_m \times j_m$ *are unrestricted, the parameter matrices,* $\Gamma_{\bar{m}\bar{m}}: l_{\bar{m}} \times j_{\bar{m}}$, $\Gamma_{m\bar{m}}: l_m \times j_{\bar{m}}$, $B_{\bar{m}\bar{m}}: j_{\bar{m}} \times j_{\bar{m}}$, $B_{\bar{m}m}: j_{\bar{m}} \times j_m$, $B_{m\bar{m}}: j_m \times j_{\bar{m}}$, $B_{\bar{m}\bar{m}}: j_{\bar{m}} \times j_{\bar{m}}$ *contain (some) parameters that are restricted to zero except for* $B_{\bar{m}\bar{m}}$, *which has all diagonal elements equal to one and some off-diagonal elements equal to zero, and* $B_{mm} = I_{j_m}$; *then the reduced form of the* SEM *from equation* (6.1) *is equal to a set of reduced rank restrictions on the standard linear model,*

$$(Y_{\bar{m}} \quad Y_m) = (Z_{\bar{m}} \quad Z_{\bar{m}m} \quad Z_m)\Phi + \xi,$$

where $\Phi: (l_{\bar{m}} + l_m + i_m) \times (j_{\bar{m}} + j_{\bar{m}})$.

Proof. See Appendix B.

Theorem 1 shows that we can use the framework for prior/posterior analysis used in the previous sections, which results from Kleibergen (1997), in a Bayesian full system analysis of a SEM. An important difference with the analysis from the previous sections is, however, the dependence of the different reduced rank restrictions on one another. For the INSEM, we can either analyze Φ conditional on (π_{11}, Π_{12}) or vice versa. So, the conditionalization of these parameters on one another does not matter. This does not hold for the full system analysis that we can conclude from the proof of Theorem 1. It results in a strict ordering in which the reduced rank restrictions have to be imposed, and hence the parameters have to be analyzed conditional on one another. The reduced form of the SEM constructed in Appendix B already shows some important conditionalization rules for the parameters of the SEM. For example, the structural form parameter $\beta_{m\bar{m}}$ is analyzed conditional on the structural form parameter $\beta_{\bar{m}m}$, which are both defined in Appendix B. More of these conditionalization rules will appear when the reduced form is constructed further.

The conditionalization rules also imply rank and order conditions that can differ from the INSEM-based conditions used in general. This is part of the point made in Maddala (1976). Regarding the conditionalization rules, the reduced form, constructed in Appendix B, shows that $\beta_{\bar{m}m}$ is identified when $\Pi_{\bar{m}\bar{m}}$ has full rank (or when that part of $\Pi_{\bar{m}\bar{m}}$ that is multiplied by the nonzero parts of $\beta_{\bar{m}m}$ has full rank), where the elements of Π are defined in Appendix B. When the INSEM-based conditions are used, it is assumed that no restrictions are imposed on $\Pi_{\bar{m}\bar{m}}$. If restrictions are imposed, however, the resulting rank and order conditions can become different. In the following, an example of this will be discussed. It can also be seen in $\beta_{m\bar{m}}$, which is identified jointly by $\Pi_{\bar{m}m}$, $\Pi_{\bar{m}\bar{m}}\beta_{\bar{m}m}$, and Π_{mm}, and its rank and order conditions therefore depend on the specification of the SEM.

As mentioned before, the framework for prior and posterior analysis used in the previous sections can also be used to construct the priors and posteriors of the parameters in a full system analysis of a SEM. When we apply this framework we have to give an exact specification of the reduced form and its (hyper) parameters reflecting the restrictions that obey the three conditions, that (i) when these (hyper) parameters are nonzero, the model is observationally equivalent with a standard linear model and when these (hyper) parameters are zero, (ii) both the reduced form of the SEM results and (iii) these (hyper) parameters are locally uncorrelated with specific other parameters such that the resulting posterior is invariant with respect to the ordering of those variables for which the likelihood is also invariant (for an exact specification of the conditions the restrictions have to satisfy, see Kleibergen, 1997). This enables us to construct the prior/posterior of the parameters of the SEM as proportional to the prior/posterior of the parameters of the linear model under the restriction that the (hyper) parameters reflecting the restrictions are zero, which is identical to the construction of priors/posteriors of the parameters of the INSEM. Because there are still some differences compared to the analysis of the INSEM, because the reduced form has a more complicated structure and the number of additional parameters we have to simulate in the posterior simulator increases (see (5.1)), we give two detailed examples, two and three (sets of) equation(s) models, to indicate all these differences. These examples jointly with Theorem 1 show how a Bayesian full system analysis of any kind of SEM is conducted.

6.1. Two (Sets of) Equations

We specify the structural form of the two (sets of) equation(s) model by

$$\begin{aligned} Y_1 &= Y_2\beta_1 + Z_1\Gamma_{11} + Z_2\Gamma_{21} + \varepsilon_1, \\ Y_2 &= Y_1\beta_2 + Z_1\Gamma_{12} + Z_3\Gamma_{32} + \varepsilon_2, \end{aligned} \tag{6.2}$$

where $Y_1{:}T \times m_1$, $Y_2{:}T \times m_2$ contain the endogenous variables; $Z_1{:}T \times k_1$, $Z_2{:}T \times k_2$, $Z_3{:}T \times k_3$ contain (weakly) exogenous and lagged dependent variables; $k_2 \geq m_1$, $k_3 \geq m_2$, $m = m_1 + m_2$, $(\varepsilon_1 \;\; \varepsilon_2) \sim n(0, \Sigma \otimes I_T)$, $\beta_1{:}m_2 \times m_1$, $\beta_2{:}m_1 \times m_2$, $\Gamma_{11}{:}k_1 \times m_1$, $\Gamma_{12}{:}k_1 \times m_2$, $\Gamma_{21}{:}k_2 \times m_1$, $\Gamma_{32}{:}k_3 \times m_2$. The reduced form of (6.2), which can be constructed using the proof of Theorem 1, reads

$$\begin{aligned} Y_1 &= Z_1\Pi_{11} + Z_2\Pi_{21} + Z_3\Pi_{32}\beta_1 + \xi_1, \\ Y_2 &= Z_1\Pi_{12} + Z_2\Pi_{21}\beta_2 + Z_3\Pi_{32} + \xi_2, \end{aligned} \tag{6.3}$$

where $\Pi_{11} = (\Gamma_{11} + \Gamma_{12}\beta_1)(I_{m_1} - \beta_2\beta_1)^{-1}$, $\Pi_{21} = \Gamma_{21}(I_{m_1} - \beta_2\beta_1)^{-1}$, $\Pi_{12} = (\Gamma_{12} + \Gamma_{11}\beta_2)(I_{m_2} - \beta_1\beta_2)^{-1}$, $\Pi_{32} = \Gamma_{32}(I_{m_2} - \beta_1\beta_2)^{-1}$, $\xi_1 = (\varepsilon_1 + \varepsilon_2\beta_1)(I_{m_1} - \beta_2\beta_1)^{-1}$, $\xi_2 = (\varepsilon_2 + \varepsilon_1\beta_2)(I_{m_2} - \beta_1\beta_2)^{-1}$, $(\xi_1 \;\; \xi_2) \sim n(0, \Omega \otimes I_T)$, $\Sigma = B'\Omega B$, and

$$B = \begin{pmatrix} I_{m_1} & -\beta_2 \\ -\beta_1 & I_{m_2} \end{pmatrix}.$$

Similar to the reduced form of the INSEM (2.2) and as indicated in the proof of Theorem 1, we add parameters to the reduced form to obtain a model, which we call unrestricted SEM (UNSEM), that is observationally equivalent with a linear model. When these added parameters are zero, both the reduced form (6.3) results and the added parameters are locally uncorrelated with specific other parameters,

$$\begin{aligned}(Y_1 \quad Y_2) &= Z_1(\Pi_{11} \quad \Pi_{12}) + Z_2\Pi_{21}\big(I_{m_1} \quad \beta_2\big) + Z_3\Pi_{32}\big(\beta_1 \quad I_{m_2}\big)\\ &\quad + Z_2\Pi_{21\perp}\lambda_2\big(I_{m_1} \quad \beta_2\big)_\perp + Z_3\Pi_{32\perp}\lambda_1\big(\beta_1 \quad I_{m_2}\big)_\perp + (\xi_1 \quad \xi_2), \qquad \textbf{(6.4)}\end{aligned}$$

where $\lambda_2 : (k_2 - m_1) \times m_2$, $\lambda_3 : (k_3 - m_2) \times m_1$ and the orthogonal complements $\Pi_{21\perp}$, $\Pi_{32\perp}$, $(I_{m_1} \ \beta_2)_\perp$, and $(\beta_1 \ I_{m_2})_\perp$ are defined similarly to the ones used in (3.1) (see Appendix C). It is clear that when $\lambda_2 = 0$, $\lambda_3 = 0$, the reduced form (6.3) results and that λ_2 and λ_3 are locally uncorrelated, when they are equal to zero, with (Π_{21}, β_2) and (Π_{32}, β_1), respectively. When $\lambda_2 \neq 0$, $\lambda_3 \neq 0$, again similar to (3.1), (6.4) is observationally equivalent with the linear model,

$$(Y_1 \quad Y_2) = (Z_1 \quad Z_2 \quad Z_3)\begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\end{pmatrix} + (\xi_1 \quad \xi_2), \qquad \textbf{(6.5)}$$

where $\Phi_1 = (\Pi_{11} \ \ \Pi_{12})$, $\Phi_2 : k_2 \times m$, $\Phi_3 : k_3 \times m$. Using a SVD, the equality of (6.4) and (6.5) can be shown. SVD's are also used to obtain $(\beta_2, \lambda_2, \Pi_{21})$ from Φ_2 and $(\beta_1, \lambda_3, \Pi_{32})$ from Φ_3 (see Appendix C). The resulting relationships are similar to (3.3)–(3.5) and straightforward to derive given (3.3)–(3.5). The SEM (6.2) is consequently a linear model with nonlinear restrictions on its parameters, $\lambda_2 = 0$, $\lambda_3 = 0$. The framework for prior/posterior analysis of the INSEM used in Sections 3 and 4 can, therefore, directly be extended to the two equation SEM (6.2). So, we specify a prior for the parameters of the linear model $(\Phi_1, \Phi_2, \Phi_3, \Omega)$, for example, a diffuse or natural conjugate prior, and this implies a prior for the parameters of the SEM (6.3) as this SEM equals the linear model evaluated in $\lambda_2 = 0$, $\lambda_3 = 0$. (Note that we use the reduced form (6.3) but this model is observationally equivalent with the SEM (6.2).)

$$\begin{aligned}&p_{sem}(\Pi_{11}, \Pi_{12}, \beta_1, \beta_2, \Pi_{21}, \Pi_{32}, \Omega)\\ &\quad\propto p_{unsem}(\Pi_{11}, \Pi_{12}, \beta_1, \beta_2, \lambda_2, \lambda_3, \Pi_{21}, \Pi_{32}, \Omega)|_{\lambda_2=0,\lambda_3=0}\\ &\quad\propto p_{lin}(\Phi_1, \Phi_2(\beta_2, \lambda_2, \Pi_{21}), \Phi_3(\beta_1, \lambda_3, \Pi_{32}), \Omega)|_{\lambda_2=0,\lambda_3=0}\\ &\quad\times |J(\Phi_2, (\beta_2, \lambda_2, \Pi_{21}))|_{\lambda_2=0}||J(\Phi_3, (\beta_1, \lambda_3, \Pi_{32}))|_{\lambda_3=0}|, \qquad \textbf{(6.6)}\end{aligned}$$

where *sem* stands for SEM, *unsem* for UNSEM, and *lin* for linear model and the Jacobians $J(\Phi_2, (\beta_2, \lambda_2, \Pi_{21}))$, $J(\Phi_3, (\beta_1, \lambda_3, \Pi_{32}))$ are straightforward to derive given the derivation of the Jacobian of the transformation in the case of the INSEM and are stated in Appendix C. Using (6.6) and the expressions of diffuse and natural conjugate priors for the linear model, (3.9) and (3.11), we can again construct the functional expressions of diffuse and natural conjugate priors for SEM's like (6.3). For reasons of compactness and similarity with Section 3 we do not give the exact functional expressions.

For the posterior exactly the same reasoning as for the prior applies, i.e., the posterior of the parameters of the SEM (6.3) is proportional to the posterior of the parameters of the linear model under the imposed restriction. We can decompose the posterior of the linear model into a product of marginal and conditional posteriors that belong to a standard class of density functions, i.e., normal or inverted-Wishart (see, e.g., Zellner, 1971). This property can directly be used to decompose the posterior of the SEM,

$$\begin{aligned}
&p_{sem}(\Pi_{11},\Pi_{12},\beta_1,\beta_2,\Pi_{21},\Pi_{32},\Omega|Y,Z)\\
&\quad\propto p_{unsem}(\Pi_{11},\Pi_{12},\beta_1,\beta_2,\lambda_2,\lambda_3,\Pi_{21},\Pi_{32},\Omega|Y,Z)|_{\lambda_2=0,\lambda_3=0}\\
&\quad\propto p_{lin}(\Phi_1,\Phi_2(\beta_2,\lambda_2,\Pi_{21}),\Phi_3(\beta_1,\lambda_3,\Pi_{32}),\Omega|Y,Z)|_{\lambda_2=0,\lambda_3=0}\\
&\qquad\times |J(\Phi_2,(\beta_2,\lambda_2,\Pi_{21}))|_{\lambda_2=0}||J(\Phi_3,(\beta_1,\lambda_3,\Pi_{32}))|_{\lambda_3=0}|\\
&\quad\propto p_{lin}(\Phi_1|\Phi_2(\beta_2,\lambda_2,\Pi_{21}),\Phi_3(\beta_1,\lambda_3,\Pi_{32}),\Omega,Y,Z)|_{\lambda_2=0,\lambda_3=0}\\
&\qquad\times p_{lin}(\Phi_2(\beta_2,\lambda_2,\Pi_{21})|\Phi_3(\beta_1,\lambda_3,\Pi_{32}),\Omega,Y,Z)|_{\lambda_2=0,\lambda_3=0}\\
&\qquad\times |J(\Phi_2,(\beta_2,\lambda_2,\Pi_{21}))|_{\lambda_2=0}|\\
&\qquad\times p_{lin}(\Phi_3(\beta_1,\lambda_3,\Pi_{32})|\Omega,Y,Z)|_{\lambda_3=0}|J(\Phi_3,(\beta_1,\lambda_3,\Pi_{32}))|_{\lambda_3=0}|\\
&\qquad\times p_{lin}(\Omega|Y,Z).
\end{aligned} \tag{6.7}$$

Note that we can also use other orderings in this decomposition. To simulate parameters from the posterior of the SEM (6.3), we use the decomposition of the posterior of the SEM (6.7). This allows us to perform the simulation in two different steps. Furthermore, we add, in each of the two different steps, parameters to the model that we, similar to Section 5, assume to be generated from some conditional density g, which we specify ourselves. In the case of diffuse priors, the following choices of these functions are natural:

$$\begin{aligned}
&g_1(\lambda_3|\beta_1,\Pi_{32},\Omega)\\
&\quad=(2\pi)^{-(1/2)l_3}|B_{1\perp}\Omega^{-1}B_{1\perp}'|^{(1/2)l_3}|\Pi_{32\perp}'Z_3'M_{(Z_1\,Z_2)}Z_3\Pi_{32\perp}|^{(1/2)m_1}\\
&\qquad\times\exp[-\tfrac{1}{2}\mathrm{tr}(B_{1\perp}\Omega^{-1}B_{1\perp}'(\lambda_3-\hat{\lambda}_3)'\Pi_{32\perp}'Z_3'M_{(Z_1\,Z_2)}Z_3\Pi_{32\perp}(\lambda_3-\hat{\lambda}_3))],
\end{aligned} \tag{6.8}$$

$$\begin{aligned}
&g_2(\lambda_2|\beta_2,\Pi_{21},\Phi_3,\Omega)\\
&\quad=(2\pi)^{-(1/2)l_2}|B_{2\perp}\Omega^{-1}B_{2\perp}'|^{(1/2)l_2}|\Pi_{21\perp}'Z_2'M_{Z_1}Z_2\Pi_{21\perp}|^{(1/2)m_2}\\
&\qquad\times\exp[-\tfrac{1}{2}\mathrm{tr}(B_{2\perp}\Omega^{-1}B_{2\perp}'(\lambda_2-\hat{\lambda}_2)'\Pi_{32\perp}'Z_2'M_{Z_1}Z_3\Pi_{32\perp}(\lambda_2-\hat{\lambda}_2))],
\end{aligned} \tag{6.9}$$

where $l_2 = k_2 - m_1$, $l_3 = k_3 - m_2$, $B_1 = (\beta_1 \ I_{m_2})$, $B_2 = (I_{m_1} \ \beta_2)$,

$$\hat{\lambda}_3 = (\Pi'_{32\perp} Z'_3 M_{(Z_1 Z_2)} Z_3 \Pi_{32\perp})^{-1} \Pi'_{32\perp} Z'_3 M_{(Z_1 Z_2)} \times (Y - Z_3 \Pi_{32} B_1) \Omega^{-1} B'_{1\perp} (B_{1\perp} \Omega^{-1} B'_{1\perp})^{-1},$$

$$\hat{\lambda}_2 = (\Pi'_{21\perp} Z'_2 M_{Z_1} Z_2 \Pi_{21\perp})^{-1} \Pi'_{21\perp} Z'_2 M_{Z_1} \times (Y - Z_3 \Phi_3 - Z_2 \Pi_{21} B_2) \Omega^{-1} B'_{2\perp} (B_{2\perp} \Omega^{-1} B'_{2\perp})^{-1}.$$

The weight functions of the two different steps of the simulation algorithm, involving both (6.8) and (6.9), then become

$$w_1(\beta_1, \lambda_3, \Pi_{32}, \Omega) = \frac{|J(\Phi_3, (\beta_1, \lambda_3, \Pi_{32}))|_{\lambda_3=0}|}{|J(\Phi_3, (\beta_1, \lambda_3, \Pi_{32}))|} g_1(\lambda_3 | \beta_1, \Pi_{32}, \Omega)|_{\lambda_3=0}, \tag{6.10}$$

$$w_2(\beta_2, \lambda_2, \Pi_{21}, \Omega | \Phi_3) = \frac{|J(\Phi_2, (\beta_2, \lambda_2, \Pi_{21}))|_{\lambda_2=0}|}{|J(\Phi_2, (\beta_2, \lambda_2, \Pi_{21})|} g_2(\lambda_2 | \beta_2, \Pi_{21}, \Phi_3, \Omega)|_{\lambda_2=0}. \tag{6.11}$$

Again these weight functions are always finite. The different steps involved in obtaining the weight attached to a certain drawing i, $i = 1, \dots, N$, of the parameters of the SEM can then be summarized as follows:

1. Draw Ω^i from $p_{lin}(\Omega | Y, Z)$.
   Draw Φ_3^i from $p_{lin}(\Phi_3 | \Omega^i, Y, Z)$.
2. Compute β_1^i, λ_3^i, Π_{32}^i from Φ_3^i using a SVD.
3. Compute $w_1(\beta_1^i, \lambda_3^i, \Pi_{32}^i, \Omega^i)$ according to (6.10).
4. Draw Φ_2^i from $p_{lin}(\Phi_2 | \Phi_3(\beta_1^i, \lambda_3, \Pi_{32}^i), \Omega^i, Y, Z)|_{\lambda_3=0}$.
5. Compute β_2^i, λ_2^i, Π_{21}^i from Φ_2^i using a SVD.
6. Compute $w_2(\beta_2^i, \lambda_2^i, \Pi_{21}^i, \Omega^i | \Phi_3(\beta_1^i, \lambda_3, \Pi_{32}^i))|_{\lambda_3=0}$ according to (6.11).
7. Compute total weight ith drawing:

   $$w(\beta_1^i, \lambda_3^i, \Pi_{32}^i, \beta_2^i, \lambda_2^i, \Pi_{21}^i, \Omega^i) = w_1 \times w_2.$$

8. Draw Φ_1^i from $p_{lin}(\Phi_1 | \Phi_2(\beta_2^i, \lambda_2, \Pi_{21}^i), \Phi_3(\beta_1^i, \lambda_3, \Pi_{32}^i), \Omega^i, Y, Z)|_{\lambda_2=0, \lambda_3=0}$.

The posteriors from which we simulate are all standard, in the case of diffuse or natural conjugate priors, and are similar to the ones used in the algorithm in Section 5. The values of other structural form parameters can be directly calculated using the equations used in (6.3) and the drawings from the preceding algorithm. The resulting total weights, w, can be used in an importance or a M-H sampler as discussed in Section 5 to obtain a posterior simulator of the posterior of the parameters of the SEM (6.3).

6.2. Three (Sets of) Equations

As an example of a three (sets of) equation(s) model, we use the following model. (Note that contrary to the two equation model, the specification of a three equation model is not unique because the model is not invariant with respect to the ordering of the variables.)

$$\begin{aligned} Y_1 &= Y_2\beta_{21} + Z_1\Gamma_{11} + \varepsilon_1, \\ Y_2 &= Y_3\beta_{32} + Z_1\Gamma_{12} + Z_2\Gamma_{22} + \varepsilon_2, \\ Y_3 &= Y_1\beta_{13} + Y_2\beta_{23} + Z_2\Gamma_{23} + Z_3\Gamma_{33} + \varepsilon_3, \end{aligned} \tag{6.12}$$

where $Y_1: T \times m_1$, $Y_2: T \times m_2$, and $Y_3: T \times m_3$ contain the endogenous variables and $Z_1: T \times k_1$, $Z_2: T \times k_2$, and $Z_3: T \times k_3$ contain lagged endogenous and weakly exogenous variables, $\beta_{21}: m_2 \times m_1$, $\beta_{32}: m_3 \times m_2$, $\beta_{13}: m_1 \times m_3$, $\beta_{23}: m_2 \times m_3$, $\Gamma_{11}: k_1 \times m_1$, $\Gamma_{12}: k_1 \times m_2$, $\Gamma_{22}: k_2 \times m_2$, $\Gamma_{23}: k_2 \times m_3$, $\Gamma_{33}: k_3 \times m_3$, $m = m_1 + m_2 + m_3$. Here $(\varepsilon_1 \ \ \varepsilon_2 \ \ \varepsilon_3) \sim n(0, \Sigma \otimes I_T)$. Because the SEM (6.12) has to be properly identified, the following (INSEM) order conditions need to be fulfilled: $k_2 + k_3 \geq m_2$, $k_3 \geq m_3$, $k_1 \geq m_1 + m_2$. Using the proof of Theorem 1, the reduced form of the model in equation (6.12) is constructed and reads

$$\begin{aligned} Y_1 &= Z_1\Pi_{11} + (Z_1 \quad Z_2\Pi_{33})\begin{pmatrix} \Pi_{22} \\ \beta_{32} \end{pmatrix}\beta_{21} + \xi_1, \\ Y_2 &= Z_1\Pi_{12} + Z_2\Pi_{22} + Z_3\Pi_{33}\beta_{32} + \xi_2, \\ Y_3 &= Z_1(\Pi_{11} \quad \Pi_{12})\begin{pmatrix} \beta_{13} \\ \beta_{23} \end{pmatrix} + Z_2\Pi_{23} + Z_3\Pi_{33} + \xi_3, \end{aligned} \tag{6.13}$$

where

$$\begin{aligned} (\Gamma_{11} \quad \Gamma_{12}) &= (\Pi_{11} \quad \Pi_{12})\begin{pmatrix} I_{m_1} & -\beta_{13}\beta_{23} \\ -\beta_{21} & I_{m_2} - \beta_{23}\beta_{13} \end{pmatrix}, \\ \Gamma_{33} &= \Pi_{33}(I_{m_3} - \beta_2(\beta_1\beta_3 - \beta_4)), \\ (\Gamma_{22} \quad \Gamma_{23}) &= (\Pi_{22} \quad \Pi_{23})\begin{pmatrix} I_{m_2} & -(\beta_{23} + \beta_{21}\beta_{13}) \\ -\beta_{32} & I_{m_3} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} (\xi_1 \quad \xi_2 \quad \xi_3)B, &= (\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3), \\ \Sigma &= B'\Omega B, \\ B &= \begin{pmatrix} I_{m_1} & 0 & -\beta_{13} \\ -\beta_{21} & I_{m_2} & -\beta_{23} \\ 0 & -\beta_{32} & I_{m_3} \end{pmatrix}. \end{aligned}$$

The reduced form (6.13) is again a system of reduced rank matrices like the reduced forms of the one equation (2.2) and two equation (6.3) models. An important difference with these models is that its reduced rank matrices depend on another, which is a.o. reflected in the identification of β_{21}, which depends on one of the other structural form parameters, β_{32}. This difference also leads to a change in the order condition compared to the INSEM. According to the INSEM order condition, β_{21} is identified when $k_2 + k_3 \geq m_2$, i.e., the number of excluded exogenous variables is at least equal to the number of included endogenous variables (see Hausman, 1983). The model (6.12) shows, however, that β_{21} is identified when $\begin{pmatrix} \Pi_{22} \\ \Pi_{33}\beta_{32} \end{pmatrix}$ has full rank. Although this matrix has $k_2 + k_3$ rows, which accords with the standard order condition, its row rank can never exceed $k_2 + m_3$ ($\leq k_2 + k_3$) as it can be specified as $\begin{pmatrix} I_{k_2} & 0 \\ 0 & \Pi_{33} \end{pmatrix}\begin{pmatrix} \Pi_{22} \\ \beta_{32} \end{pmatrix}$ and the last matrix in this product has $k_2 + m_3$ rows. It is therefore important that the identification of the different parameters of a SEM in a full system analysis is conducted using the restricted reduced form parameter matrix instead of the unrestricted one as this can lead to different rank and order conditions (see also Maddala, 1976). This different order condition results from the dependence on one another of the reduced rank structures imposed by the SEM (6.13) (see also proof of Theorem 1). The reduced rank structures appearing in the two equation model do not depend on one another, as can be concluded from (6.4), and therefore the INSEM order conditions still apply there.

As a consequence of the sequential dependence between the reduced rank structures, not only do the order conditions of the INSEM and the SEM (6.12) differ, as indicated previously, but also the parameters that we add to the model (6.13), to make it observationally equivalent to a linear model, are different from the ones we used before (see also the proof of Theorem 1). In the cases of the INSEM (3.1) and the two equation SEM (6.4), the parameters added to the reduced form, to make it observationally equivalent to a linear model, do not depend on one another in a sequential way. The parameters added to (6.13) do, however, depend on each other sequentially. To show this, consider the linear model

$$(Y_1 \quad Y_2 \quad Y_3) = (Z_1 \quad Z_2 \quad Z_3)\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} + (\xi_1 \quad \xi_2 \quad \xi_3). \tag{6.14}$$

The reduced form model (6.13) can be obtained by using what we call an unrestricted SEM specification of the parameters of (6.14),

$$\Phi_1 = (\Pi_{11} \quad \Pi_{12})\begin{pmatrix} I_{m_1} & 0 & \beta_{13} \\ 0 & I_{m_2} & \beta_{23} \end{pmatrix} + (\Pi_{11} \quad \Pi_{12})_\perp \lambda_1 \begin{pmatrix} I_{m_1} & 0 & \beta_{13} \\ 0 & I_{m_2} & \beta_{23} \end{pmatrix}_\perp, \tag{6.15}$$

$$\begin{pmatrix} \Phi_2 \\ \Phi_3 \end{pmatrix} = \Theta \begin{pmatrix} \beta_{21} & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{pmatrix} + \Theta_\perp \lambda_2 \begin{pmatrix} \beta_{21} & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{pmatrix}_\perp, \tag{6.16}$$

$$\Theta = \begin{pmatrix} \Pi_{22} & \Pi_{23} \\ & \Theta_2 \end{pmatrix}, \tag{6.17}$$

$$\Theta_2 = \Pi_{33}(\beta_{32} \quad I_{m_3}) + \Pi_{33\perp}\lambda_3(\beta_{32} \quad I_{m_3})_\perp, \tag{6.18}$$

where the orthogonal complements are defined similarly to the ones used in (3.1) (see also Appendix D), $\lambda_1 : (k_1 - m_1 - m_2) \times m_3$, $\lambda_2 : (k_2 + k_3 - m_2 - m_3) \times m_1$, $\lambda_3 : (k_3 - m_3) \times m_2$. To analyze the implications of the different orthogonality conditions in (6.15)–(6.18), we substitute the expression of Θ in $(\Phi_2' \;\; \Phi_3')'$,

$$\begin{pmatrix} \Phi_2 \\ \Phi_3 \end{pmatrix} = \begin{pmatrix} \Pi_{22} & \Pi_{23} \\ \Pi_{33}(\beta_{32} \;\; I_{m_3}) + \Pi_{33\perp}\lambda_3(\beta_{32} \;\; I_{m_3})_\perp \end{pmatrix} \begin{pmatrix} \beta_{21} & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{pmatrix} + \begin{pmatrix} \Pi_{22} & \Pi_{23} \\ \Pi_{33}(\beta_{32} \;\; I_{m_3}) + \Pi_{33\perp}\lambda_3(\beta_{32} \;\; I_{m_3})_\perp \end{pmatrix}_\perp \lambda_2 \begin{pmatrix} \beta_{21} & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{pmatrix}_\perp . \tag{6.19}$$

It is clear from (6.15)–(6.18) that when $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$, the model (6.13) results. Furthermore, when $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$, λ_1 is locally uncorrelated with $(\Pi_{11}, \Pi_{12}, \beta_{13}, \beta_{23})$, λ_3 with (Π_{33}, β_{32}), and λ_2 with β_{21} and all parameters contained in Θ, i.e., Π_{22}, Π_{23}, Π_{33}, λ_3, β_{32}. SVD's are needed to obtain $(\Pi_{11}, \Pi_{12}, \lambda_1, \beta_{13}, \beta_{23})$ from Φ_1, $(\Theta, \lambda_2, \beta_{21})$ from (Φ_2, Φ_3), and $(\Pi_{22}, \Pi_{23}, \Pi_{33}, \lambda_3, \beta_{32})$ from Θ and to show the observational equivalence between the model imposed by (6.15)–(6.18) and (6.14) when $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$. These SVD's are stated in Appendix D. The sequential dependence between the structural form parameters now reflects itself in SVD's that have to be applied recursively, a.o. of Θ, which already results from a SVD as it is the reduced form of an INSEM,

$$\begin{aligned} \tilde{Y}_1 &= \tilde{Y}_2 \beta_{32} + \tilde{Z}_1 \Delta_{11} + v_1, \\ \tilde{Y}_2 &= \tilde{Z}_1 \Delta_{21} + \tilde{Z}_2 \Delta_{22} + v_2, \end{aligned} \tag{6.20}$$

where $\tilde{Y}$, $\tilde{Y}_2$, $\tilde{Z}_1$, and $\tilde{Z}_2$ are data matrices, $\Delta_{21} = \Pi_{23}$, $\Delta_{22} = \Pi_{33}$, $\Delta_{11} = \Pi_{22} - \Delta_{21}\beta_{32}$. Therefore Θ is similar to the $(\Pi_{\bar{m}\bar{m}}' \;\; \Pi_{m\bar{m}}')'$ parameter matrix used in the proof of Theorem 1.

So, the SEM (6.13) is again a linear model with restrictions on its parameters. We can, therefore, again apply the framework for prior/posterior analysis used in the previous sections, i.e., we specify the prior/posterior of the parameters of (6.13) as proportional to the prior/posterior of the parameters of the linear model under the condition that the restrictions hold,

$$\begin{aligned} &p_{sem}(\beta_{21}, \beta_{32}, \beta_{13}, \beta_{23}, \Pi_{11}, \Pi_{12}, \Pi_{22}, \Pi_{23}, \Pi_{33}, \Omega) \\ &\quad \propto p_{unsem}(\beta_{21}, \beta_{32}, \beta_{13}, \beta_{23}, \lambda_1, \lambda_2, \lambda_3, \Pi_{11}, \Pi_{12}, \Pi_{22}, \Pi_{23}, \Pi_{33}, \Omega)|_{(\lambda_1, \lambda_2, \lambda_3) = 0} \\ &\quad \propto p_{lin}(\Phi_1(\beta_{13}, \beta_{23}, \lambda_1, \Pi_{11}, \Pi_{12}), \\ &\qquad (\Phi_2, \Phi_3)(\beta_{21}, \lambda_2, \Theta(\Pi_{22}, \Pi_{23}, \Pi_{33}, \lambda_3, \beta_{32}), \Omega)|_{(\lambda_1, \lambda_2, \lambda_3) = 0} \\ &\qquad \times |J(\Phi_1, (\beta_{13}, \beta_{23}, \lambda_1, \Pi_{11}, \Pi_{12}))|_{\lambda_1 = 0}| \\ &\qquad \times |J(\Theta, (\Pi_{22}, \Pi_{23}, \Pi_{33}, \lambda_3, \beta_{32}))|_{\lambda_3 = 0}| \\ &\qquad \times |J((\Phi_2, \Phi_3), (\beta_{21}, \lambda_2, \Theta(\Pi_{22}, \Pi_{23}, \Pi_{33}, \lambda_3, \beta_{32})|_{\lambda_3 = 0}))|_{\lambda_2 = 0}|, \end{aligned} \tag{6.21}$$

where $J(\Phi_1,(\beta_{13},\beta_{23},\lambda_1,\Pi_{11},\Pi_{12}))$, $J(\Theta,(\Pi_{22},\Pi_{23},\Pi_{33},\lambda_3,\beta_{32}))$, $J((\Phi_2,\Phi_3),(\beta_{21},\lambda_2,\Theta))$ are the Jacobians of the transformation from Φ_1 to $(\beta_{13},\beta_{23},\lambda_1,\Pi_{11},\Pi_{12})$, (Φ_2,Φ_3) to $(\beta_{21},\lambda_2,\Theta)$ and Θ to $(\Pi_{22},\Pi_{23},\Pi_{33},\lambda_3,\beta_{32})$. These Jacobians are stated in Appendix D. When we specify a diffuse (3.9) or natural conjugate prior (3.11) for the parameters of the linear model, (6.21) shows the implied prior for the parameters of the SEM. We do not give the exact functional expressions as they can be constructed along the lines of Section 3.

Also for the posterior, we use the framework from Kleibergen (1997). Furthermore, we use the decomposition of the posterior of the linear model into a product of conditional and marginal densities,

$$
\begin{aligned}
&p_{sem}(\beta_{21},\beta_{32},\beta_{13},\beta_{23},\Pi_{11},\Pi_{12},\Pi_{22},\Pi_{23},\Pi_{33},\Omega|Y,Z)\\
&\quad\propto p_{unsem}(\beta_{21},\beta_{32},\beta_{13},\beta_{23},\lambda_1,\lambda_2,\lambda_3,\Pi_{11},\\
&\qquad\qquad \Pi_{12},\Pi_{22},\Pi_{23},\Pi_{33},\Omega|Y,Z)|_{(\lambda_1,\lambda_2,\lambda_3)=0}\\
&\quad\propto p_{lin}(\Phi_1(\beta_{13},\beta_{23},\lambda_1,\Pi_{11},\Pi_{12}),\\
&\qquad (\Phi_2,\Phi_3)(\beta_{21},\lambda_2,\Theta(\Pi_{22},\Pi_{23},\Pi_{33},\lambda_3,\beta_{32}),\Omega|Y,Z)|_{(\lambda_1,\lambda_2,\lambda_3)=0}\\
&\qquad \times|J(\Phi_1,(\beta_{13},\beta_{23},\lambda_1,\Pi_{11},\Pi_{12}))|_{\lambda_1=0}|\\
&\qquad \times|J(\Theta,(\Pi_{22},\Pi_{23},\Pi_{33},\lambda_3,\beta_{32}))|_{\lambda_3=0}|\\
&\qquad \times|J((\Phi_2,\Phi_3),(\beta_{21},\lambda_2,\Theta(\Pi_{22},\Pi_{23},\Pi_{33},\lambda_3,\beta_{32})|_{\lambda_3=0}))|_{\lambda_2=0}|,\\
&\quad\propto p_{lin}(\Phi_1(\beta_{13},\beta_{23},\lambda_1,\Pi_{11},\Pi_{12})|(\Phi_2,\Phi_3)(\beta_{21},\lambda_2,\Theta),\Omega|Y,Z)|_{(\lambda_1,\lambda_2,\lambda_3)=0}\\
&\qquad \times|J(\Phi_1,(\beta_{13},\beta_{23},\lambda_1,\Pi_{11},\Pi_{12}))|_{\lambda_1=0}|\\
&\qquad \times p_{lin}((\Phi_2,\Phi_3)(\beta_{21},\lambda_2,\Theta(\Pi_{22},\Pi_{23},\Pi_{33},\lambda_3,\beta_{32})|\Omega,Y,Z)|_{(\lambda_2,\lambda_3)=0}\\
&\qquad \times|J((\Phi_2,\Phi_3),(\beta_{21},\lambda_2,\Theta(\Pi_{22},\Pi_{23},\Pi_{33},\lambda_3,\beta_{32})|_{\lambda_3=0}))|_{\lambda_2=0}|\\
&\qquad \times|J(\Theta,(\Pi_{22},\Pi_{23},\Pi_{33},\lambda_3,\beta_{32}))|_{\lambda_3=0}|p_{lin}(\Omega|Y,Z).
\end{aligned}
\tag{6.22}
$$

We note that for this model only a few decompositions of the posterior into conditional and marginal posteriors are allowed for, i.e., (Φ_2,Φ_3) given Φ_1 and vice versa, because of the reduced rank structure imposed by the SEM. We cannot for example analyze Φ_2 given Φ_3 or vice versa. We use the decomposition of the posterior (6.22) to construct a posterior simulator. Again, similar to previous sections, to simulate from the posterior of (6.13) we add parameters to the model, i.e., $\lambda_1,\lambda_2,\lambda_3$, which we assume to be drawn from a specific conditional density, which we specify ourselves (see (5.1)). In the case of a diffuse prior for the linear model (3.9), natural choices for these conditional densities are

$$g_1(\lambda_1|\beta_{13},\beta_{23},\Pi_{11},\Pi_{12},\Phi_2,\Phi_3,\Omega)$$
$$= (2\pi)^{-(1/2)l_1}|B_{1\perp}\Omega^{-1}B'_{1\perp}|^{(1/2)l_1}|(\Pi_{11}\quad \Pi_{12})'_{\perp}Z'_1Z_1(\Pi_{11}\quad \Pi_{12})_{\perp}|^{(1/2)m_3}$$
$$\times \exp[-\tfrac{1}{2}\text{tr}(B_{1\perp}\Omega^{-1}B'_{1\perp}(\lambda_1-\hat{\lambda}_1)'$$
$$\times (\Pi_{11}\quad \Pi_{12})'_{\perp}Z'_1Z_1(\Pi_{11}\quad \Pi_{12})_{\perp}(\lambda_1-\hat{\lambda}_1))], \tag{6.23}$$

$$g_2(\lambda_2|\beta_{21},\Theta,\Omega)$$
$$= (2\pi)^{-(1/2)l_2}|B_{2\perp}\Omega^{-1}B'_{2\perp}|^{(1/2)l_2}|\Theta'_{\perp}(Z_2\quad Z_3)'M_{Z_1}(Z_2\quad Z_3)\Theta_{\perp}|^{(1/2)m_1}$$
$$\times \exp[-\tfrac{1}{2}\text{tr}(B_{2\perp}\Omega^{-1}B'_{2\perp}(\lambda_2-\hat{\lambda}_2)'\Theta'_{\perp}$$
$$\times (Z_2\quad Z_3)'M_{Z_1}(Z_2\quad Z_3)\Theta_{\perp}(\lambda-\hat{\lambda}_2))], \tag{6.24}$$

$$g_3(\lambda_3|\beta_{32},\beta_{21},\Pi_{33},\Omega)$$
$$= (2\pi)^{-(1/2)l_3}|B_{3\perp}B_2\Omega^{-1}B'_2B'_{3\perp}|^{(1/2)l_3}|\Pi'_{33\perp}Z'_3M_{(Z_1\,Z_2)}Z_3)\Pi_{33\perp}|^{(1/2)m_2}$$
$$\times \exp[-\tfrac{1}{2}\text{tr}(B_{3\perp}B_2\Omega^{-1}B'_2B'_{3\perp}(\lambda_3-\hat{\lambda}_3)'\Pi'_{33\perp}$$
$$\times Z'_3M_{(Z_1\,Z_2)}Z_3)\Pi_{33\perp}(\lambda_3-\hat{\lambda}_3))], \tag{6.25}$$

where $l_1 = k_1 - m_1 - m_2$, $l_2 = k_2 + k_3 - m_2 - m_3$, $l_3 = k_3 - m_3$,

$$B_1 = \begin{pmatrix} I_{m_1+m_2} & \begin{pmatrix}\beta_{13}\\ \beta_{23}\end{pmatrix}\end{pmatrix},$$

$$B_2 = \begin{pmatrix} \begin{pmatrix}\beta_{21}\\ 0\end{pmatrix} & I_{m_2+m_3}\end{pmatrix},$$

$$B_3 = \begin{pmatrix}\beta_{32} & I_{m_3}\end{pmatrix},$$

$$\hat{\lambda}_1 = ((\Pi_{11}\quad \Pi_{12})'_{\perp}Z'_1Z_1(\Pi_{11}\quad \Pi_{12})_{\perp})^{-1}(\Pi_{11}\quad \Pi_{12})'_{\perp}Z'_1$$
$$\times (Y - Z_2\Phi_2 - Z_3\Phi_3 - Z_1(\Pi_{11}\quad \Pi_{12})B_1)\Omega^{-1}B'_{1\perp}(B_{1\perp}\Omega^{-1}B'_{1\perp})^{-1},$$

$$\hat{\lambda}_2 = (\Theta'_{\perp}(Z_2\quad Z_3)'M_{Z_1}(Z_2\quad Z_3)\Theta_{\perp})^{-1}\Theta'_{\perp}(Z_2\quad Z_3)'M_{Z_1}$$
$$\times (Y - (Z_2\quad Z_3)\Theta B_2)\Omega^{-1}B'_{2\perp}(B_{2\perp}\Omega^{-1}B'_{2\perp})^{-1},$$

$$\hat{\lambda}_3 = \left(\Pi'_{33\perp}Z'_3M_{(Z_1\,Z_2)}Z_3\quad \Pi_{33\perp}\right)^{-1}\Pi'_{33\perp}Z'_3M_{(Z_1\,Z_2)}$$
$$\times (Y - Z_3\Pi_{33}B_3B_2)\Omega^{-1}B'_2B'_{3\perp}(B_{3\perp}B_2\Omega^{-1}B'_2B'_{3\perp})^{-1}.$$

Because we simulate from a density that approximates the posterior of (6.13), weight functions are involved in the different steps of the posterior simulator. As we simulate three different parameters, i.e., $\lambda_1,\lambda_2,\lambda_3$, that are not present in the original posterior we want to simulate from, three weight functions are involved,

$$w_1(\beta_{13},\beta_{23},\lambda_1,\Pi_{11},\Pi_{12},\Omega|\Phi_2,\Phi_3)$$
$$= \frac{|J(\Phi_1,(\beta_{13},\beta_{23},\lambda_1,\Pi_{11},\Pi_{12}))|_{\lambda_1=0}|}{|J(\Phi_1,(\beta_{13},\beta_{23},\lambda_1,\Pi_{11},\Pi_{12}))|}$$
$$\times g_1(\lambda_1|\beta_{13},\beta_{23},\Pi_{11},\Pi_{12},\Phi_2,\Phi_3,\Omega)|_{\lambda_1=0} \quad \textbf{(6.26)}$$

$$w_2(\beta_{21},\lambda_2,\Theta,\Omega)$$
$$= \frac{|J((\Phi_2,\Phi_3),(\beta_{21},\lambda_2,\Theta))|_{\lambda_2=0,\lambda_3=0}|}{|J((\Phi_2,\Phi_3),(\beta_{21},\lambda_2,\Theta))|_{\lambda_3\neq 0}|} g_2(\lambda_2|\beta_{21},\Theta,\Omega)|_{\lambda_2=0,\lambda_3=0} \quad \textbf{(6.27)}$$

$$w_3(\beta_{32},\beta_{21},\lambda_3,\Pi_{33},\Omega)$$
$$= \frac{|J(\Theta_2,(\beta_{32},\lambda_3,\Pi_{33}))|_{\lambda_2=0}|}{|J(\Theta_2,(\beta_{32},\lambda_3,\Pi_{33}))|} g_3(\lambda_3|\beta_{32},\beta_{21},\Pi_{33},\Omega)|_{\lambda_3=0}, \quad \textbf{(6.28)}$$

where $J(\Phi_1,(\beta_{13},\beta_{23},\lambda_1,\Pi_{11},\Pi_{12}))$, $J((\Phi_2,\Phi_3),(\beta_{21},\lambda_2,\Theta))$, and $J(\Theta_2,(\beta_{32},\lambda_3,\Pi_{33}))$ are the Jacobians of the different parameter transformations and each of the weight functions is always finite (see Appendix D).

The different steps involved in obtaining the weight attached to a certain drawing $i, i=1,\dots,N$, of the parameters of the SEM (6.13) can then be summarized as follows:

1. Draw Ω^i from $p_{lin}(\Omega|Y,Z)$.
   Draw (Φ_2^i,Φ_3^i) from $p_{lin}(\Phi_2,\Phi_3|\Omega^i,Y,Z)$.
2. Compute $\beta_{21}^i,\lambda_2^i,\Theta^i$ from (Φ_2^i,Φ_3^i) using SVD.
3. Compute $\beta_{32}^i,\lambda_3^i,\Pi_{33}^i$ from Θ_2^i using SVD.
4. Compute $w_3(\beta_{32}^i,\beta_{21}^i,\lambda_3^i,\Pi_{33}^i,\Omega^i)$.
5. Compute $w_2(\beta_{21}^i,\lambda_2^i,\Theta^i,\Omega^i)$.
6. Draw Φ_1^i from $p_{lin}(\Phi_1|\Phi_2^i,\Phi_3^i,\Omega^i,Y,Z)|_{\lambda_2=0,\lambda_3=0}$.
7. Compute $\beta_{13}^i,\beta_{23}^i,\lambda_1^i,\Pi_{11}^i,\Pi_{12}^i$ from Φ_1^i.
8. Compute $w_1(\beta_{13}^i,\beta_{23}^i,\lambda_1^i,\Pi_{11}^i,\Pi_{12}^i,\Omega^i|\Phi_2^i,\Phi_3^i)|_{\lambda_2=0,\lambda_3=0}$.
9. Compute total weight ith drawing: $w = w_1 \times w_2 \times w_3$.

The total weights can be used in an importance or an M-H sampler, as indicated in Section 5, to obtain a posterior simulator of the posterior of the parameters of (6.13).

The means of the conditional posteriors of Φ_1 given (Φ_2,Φ_3) and (Φ_2,Φ_3) given Φ_1 can also be used in an iterative scheme to obtain the full information maximum likelihood estimator of $(\beta_{21},\beta_{32},\beta_{13},\beta_{23},\Pi_{11},\Pi_{12},\Pi_{22},\Pi_{23},\Pi_{33})$ (see Hausman, 1983). This is similar to the INSEM where evaluating the posterior of Φ at its posterior mean using a diffuse prior gives a similar analytical expression as the limited information maximum likelihood estimator of β and Π_{22} when using the involved SVD. The iterative scheme for obtaining the full information maximum likelihood estimator proceeds as follows:

(0) Initialize $\Phi_1 = \hat{\Phi}_1$;
(i) Construct $(\beta_{21}, \beta_{32}, \Pi_{22}, \Pi_{23}, \Pi_{33})$ from $(\tilde{\Phi}_2, \tilde{\Phi}_3)$ using SVD's from steps 2 and 3 from the simulation scheme;
(ii) Compute value of (Φ_2, Φ_3) implied by $(\beta_{21}, \beta_{32}, \Pi_{22}, \Pi_{23}, \Pi_{33})$;
(iii) Construct $(\beta_{13}, \beta_{23}, \Pi_{11}, \Pi_{12})$ from $\tilde{\Phi}_1$ using SVD from step 7;
(iv) Compute value of Φ_1 implied by $(\beta_{13}, \beta_{23}, \Pi_{11}, \Pi_{12})$;
(v) Unless (Φ_1, Φ_2, Φ_3) have converged goto (i)

where $(\tilde{\Phi}_2' \ \tilde{\Phi}_3')' = ((Z_2 \ Z_3)'(Z_2 \ Z_3))^{-1}(Z_2 \ Z_3)'(Y - Z_1\Phi_1)$, $\tilde{\Phi}_1 = (Z_1'Z_1)^{-1} \times Z_1'(Y - (Z_2 \ Z_3)(\Phi_2' \ \Phi_3')'$. Using Theorem 1, iterative schemes similar to the preceding one can be constructed to obtain the full information maximum likelihood estimators of the parameters of generally specified SEM's. Jointly with the examples of the two and three structural equations SEM's, Theorem 1 shows how Bayesian analyses of generally specified SEM's are conducted.

7. CONCLUSIONS

The traditional Bayesian analyses of SEM's using diffuse priors, as proposed by, for example, Drèze (1976), Drèze and Morales (1976), and Drèze and Richard (1983), suffer from local nonidentification problems that lead to an a posteriori favor for certain parameter values that is not the result of information in the prior or data. We therefore use a framework constructed in Kleibergen (1997) in which the priors/posteriors of the parameters of the SEM are proportional to the priors/posteriors of the parameters of a linear model under the condition that the restrictions, imposed by the SEM on the parameters of the linear model, hold. We applied this framework to examples of one, two, and three structural equation SEM's, for which expressions of the priors and posteriors are derived jointly with posterior simulators. Using a theorem that states that the reduced form of any kind of SEM accords with a linear model with reduced rank restrictions of its parameters, the analysis of the examples can be generalized to other specifications of SEM's in a straightforward way. This theorem also shows how full information maximum likelihood estimators can be constructed.

Using results from Kleibergen and Paap (1997), we can also construct tools for model comparison like Bayes factors, posterior odds ratios, and Bayesian Lagrange multiplier statistics. In future work we will construct and apply these procedures to analyze the support for (multiple structural equations) SEM's in practice. It is also interesting to analyze the theoretical properties of the derived posteriors, as for example in Chao and Phillips (1998), where functional expressions are constructed for the marginal posterior of the structural form parameters of the INSEM using a Jeffrey's prior, to investigate the similarities/differences between small sample distributions of classical statistical estimators and the marginal posteriors of the structural form parameters (see Kleibergen and Zivot, 1998). Both limited information maximum likelihood (LIML) estimators (see Anderson and Rubin, 1949) and the posteriors of the parameters of the INSEM are namely constructed using SVD's (see Kleibergen and Zivot, 1998) that correspond with canonical correlations in the case of the LIML estimator. So, it is interesting to investigate to what extent these similarities hold further.

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APPENDIX A. JACOBIAN OF TRANSFORMATION FROM LINEAR MODEL TO INSEM

For the derivation of the Jacobian of the transformation from the linear model parameters to the parameters of the INSEM, it is notationally convenient to conduct this transformation in two steps: (i) from Φ to $(\Pi_{221}, \theta_2, \beta, \lambda)$ where $\theta_2 = \Pi_{222}\Pi_{221}^{-1}$ and (ii) from $(\Pi_{221}, \theta_2, \beta, \lambda)$ to $(\Pi_{221}, \Pi_{222}, \beta, \lambda)$. In the following we construct the Jacobians of the two transformations.

We can denote Φ as

$$\Phi = (\theta \quad \theta_\perp)\begin{pmatrix}\Pi_{221} & 0\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}B\\ B_\perp\end{pmatrix}$$

$$= \theta\Pi_{221}B + \theta_\perp \lambda B_\perp,$$

where $\theta = (I_{m-1} \ \theta_2')'$, $\theta_\perp = (-\theta_2 \ I_{k_2-m+1})'(I_{k_2-m+1} + \theta_2\theta_2')^{-1/2}$, $B = (\beta \ I_{m-1})$, $B_\perp = (1+\beta'\beta)^{-1/2}(1 \ -\beta')$. The Jacobians of Φ with respect to Π_{221}, θ_2, β, and λ then read

$$J_1 = \frac{\partial \operatorname{vec}(\Phi)}{\partial \operatorname{vec}(\Pi_{221})'} = (B' \otimes \theta),$$

$$J_2 = \frac{\partial \operatorname{vec}(\Phi)}{\partial \operatorname{vec}(\theta_2)'} = (B'\Pi_{221}' \otimes I_{k_2})\frac{\partial \operatorname{vec}(\theta)}{\partial \operatorname{vec}(\theta_2)'} + (B_\perp'\lambda' \otimes I_{k_2})\frac{\partial \operatorname{vec}(\theta_\perp)}{\partial \operatorname{vec}(\theta_2)'},$$

$$J_3 = \frac{\partial \operatorname{vec}(\Phi)}{\partial \operatorname{vec}(\beta)'} = (I_m \otimes \theta\Pi_{221})\frac{\partial \operatorname{vec}(B)}{\partial \operatorname{vec}(\beta)'} + (I_m \otimes \theta_\perp\lambda)\frac{\partial \operatorname{vec}(B_\perp)}{\partial \operatorname{vec}(\beta)'},$$

$$J_4 = \frac{\partial \operatorname{vec}(\Phi)}{\partial \operatorname{vec}(\lambda)'} = (B_\perp' \otimes \theta_\perp),$$

where

$$\frac{\partial \operatorname{vec}(\theta)}{\partial \operatorname{vec}(\theta_2)'} = \left(I_{m-1} \otimes \begin{pmatrix}0\\ I_{k_2-m+1}\end{pmatrix}\right),$$

$$\frac{\partial \operatorname{vec}(\theta_\perp)}{\partial \operatorname{vec}(\theta_2)'} = -\left(H^{-(1/2)'} \otimes \begin{pmatrix}I_{m-1}\\ 0\end{pmatrix}\right)K_{k_2-m+1,m-1}$$

$$+ \left(I_{k_2-m+1} \otimes \begin{pmatrix}-\theta_2'\\ I_{k_2-m+1}\end{pmatrix}\right)\frac{\partial \operatorname{vec}((H^{1/2})^{-1})}{\partial \operatorname{vec}(H^{1/2})'}\frac{\partial \operatorname{vec}(H^{1/2})}{\partial \operatorname{vec}(H)'}\frac{\partial \operatorname{vec}(H)}{\partial \operatorname{vec}(\theta_2)'},$$

$$\frac{\partial \operatorname{vec}((H^{1/2})^{-1})}{\partial \operatorname{vec}(H^{1/2})'} = -(H^{-(1/2)'} \otimes H^{-1/2}),$$

$$\frac{\partial \operatorname{vec}(H^{1/2})}{\partial \operatorname{vec}(H)'} = ((I_{k_2-m+1} \otimes H^{1/2}) + (H^{(1/2)'} \otimes I_{k_2-m+1}))^{-1},$$

$$\frac{\partial \operatorname{vec}(H)}{\partial \operatorname{vec}(\theta_2)'} = (\theta_2 \otimes I_{k_2-m+1}) + (I_{k_2-m+1} \otimes \theta_2)K_{k_2-m+1,m-1},$$

$$\frac{\partial \operatorname{vec}(B)}{\partial \operatorname{vec}(\beta)'} = (e_1 \otimes I_{m-1}),$$

$$\frac{\partial \operatorname{vec}(B_\perp)}{\partial \operatorname{vec}(\beta)'} = -\left(\begin{pmatrix} 0 \\ I_{m-1} \end{pmatrix} \otimes \mathcal{B}^{-1/2}\right) K_{m-1,1}$$

$$+ ((1 - \beta')' \otimes 1) \frac{\partial \operatorname{vec}(\mathcal{B}^{-1/2})}{\partial \operatorname{vec}(\mathcal{B}^{1/2})'} \frac{\partial \operatorname{vec}(\mathcal{B}^{1/2})}{\partial \operatorname{vec}(\mathcal{B})'} \frac{\partial \operatorname{vec}(\mathcal{B})}{\partial \operatorname{vec}(\beta)'},$$

$$\frac{\partial \operatorname{vec}(\mathcal{B}^{-1/2})}{\partial \operatorname{vec}(\mathcal{B}^{1/2})'} = -(\mathcal{B}^{-(1/2)'} \otimes \mathcal{B}^{-1/2}) = -\mathcal{B}^{-1},$$

$$\frac{\partial \operatorname{vec}(\mathcal{B}^{1/2})}{\partial \operatorname{vec}(\mathcal{B})'} = ((1 \otimes \mathcal{B}^{1/2}) + (\mathcal{B}^{(1/2)'} \otimes 1))^{-1} = \frac{1}{2}\mathcal{B}^{-1/2},$$

$$\frac{\partial \operatorname{vec}(\mathcal{B}^{1/2})}{\partial \operatorname{vec}(\beta)'} = (\beta' \otimes 1)K_{m-1,1} + (1 \otimes \beta') = 2\beta'$$

and $H = I_{k_2-m+1} + \theta_2\theta_2'$, $H^{1/2}H^{1/2} = H$, $\mathcal{B} = (1 + \beta'\beta)$, $\mathcal{B}^{1/2}\mathcal{B}^{1/2} = \mathcal{B}$, e_1 is the first m-dimensional unity vector, $K_{i,j}: ij \times ij$ are so-called commutation matrices such that for any $W: i \times j$, $\operatorname{vec}(W') = K_{i,j}\operatorname{vec}(W)$, $\operatorname{vec}(W) = K_{j,i}\operatorname{vec}(W')$, $K_{j,i} = K_{i,j}'$ (see Magnus and Neudecker, 1988). Note that when Q is symmetric, $Q = P\Lambda P'$, where P are orthogonal eigenvectors and Λ is a diagonal matrix containing the eigenvalues, then $Q^{1/2} = P\Lambda^{1/2}P'$ is also symmetric.

The Jacobian of the transformation from Φ to $(\Pi_{221}, \theta_2, \beta, \lambda)$ then reads

$$\frac{\partial \operatorname{vec}(\Phi)}{\partial(\operatorname{vec}(\Pi_{221})'\operatorname{vec}(\theta_2)'\operatorname{vec}(\beta)'\operatorname{vec}(\lambda)')} = (J_1 \quad J_2 \quad J_3 \quad J_4).$$

Because $\theta_2 = \Pi_{222}\Pi_{221}^{-1}$, the Jacobians of the transformations from $(\Pi_{221}, \theta_2, \beta, \lambda)$ to Π_{221}, Π_{222}, β, and λ read

$$G_1 = \frac{\partial(\operatorname{vec}(\Pi_{221})'\operatorname{vec}(\theta_2)'\operatorname{vec}(\beta)'\operatorname{vec}(\lambda)')'}{\partial \operatorname{vec}(\Pi_{221})'} = \begin{pmatrix} I_{m-1} \otimes I_{m-1} \\ -\Pi_{221}^{-1\prime} \otimes \Pi_{222}\Pi_{221}^{-1} \\ 0 \\ 0 \end{pmatrix},$$

$$G_2 = \frac{\partial(\operatorname{vec}(\Pi_{221})'\operatorname{vec}(\theta_2)'\operatorname{vec}(\beta)'\operatorname{vec}(\lambda)')'}{\partial \operatorname{vec}(\Pi_{222})'} = \begin{pmatrix} 0 \\ \Pi_{221}^{-1\prime} \otimes I_{k_2-m+1} \\ 0 \\ 0 \end{pmatrix},$$

$$G_3 = \frac{\partial(\operatorname{vec}(\Pi_{221})'\operatorname{vec}(\theta_2)'\operatorname{vec}(\beta)'\operatorname{vec}(\lambda)')'}{\partial \operatorname{vec}(\beta)'} = \begin{pmatrix} 0 \\ 0 \\ 1 \otimes I_{m-1} \\ 0 \end{pmatrix},$$

$$G_4 = \frac{\partial(\text{vec}(\Pi_{221})'\ \text{vec}(\theta_2)'\ \text{vec}(\beta)'\ \text{vec}(\lambda)')'}{\partial\,\text{vec}(\lambda)'} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \otimes I_{k_2-m+1} \end{pmatrix}.$$

The Jacobian of the transformation from Φ to $(\Pi_{22}, \beta, \lambda)$ then becomes

$$\begin{aligned}
|J(\Phi,(\Pi_{22},\beta,\lambda))| &= \left| \frac{\partial\,\text{vec}(\Phi)}{\partial(\text{vec}(\Pi_{22})'\ \text{vec}(\beta)'\text{vec}(\lambda)')} \right| \\
&= \left| \frac{\partial\,\text{vec}(\Phi)}{\partial(\text{vec}(\Pi_{221})'\text{vec}(\theta_2)'\ \text{vec}(\beta)'\ \text{vec}(\lambda)')} \right| \\
&\quad \times \left| \frac{\partial(\text{vec}(\Pi_{221})'\ \text{vec}(\theta_2)'\ \text{vec}(\beta)'\ \text{vec}(\lambda)')'}{\partial(\text{vec}(\Pi_{22})'\ \text{vec}(\beta)'\ \text{vec}(\lambda)')} \right| \\
&= |(J_1 \quad J_2 \quad J_3 \quad J_4)||(G_1 \quad G_2 \quad G_3 \quad G_4)|.
\end{aligned}$$

So,

$$J(\Phi,(\Pi_{22},\beta,\lambda))|_{\lambda=0} = \left(B' \otimes I_{k_2} \quad e_1 \otimes \Pi_{22} \quad B'_\perp \otimes \Pi_{22\perp}\right).$$

To prove that $|(J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta))| \geq |(J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta))|_{\lambda=0})|$, we use that

$$J(\Phi,(\Pi_{22},\lambda,\beta)) = J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta))J((\Pi_{221},\theta_2,\lambda,\beta),(\Pi_{22},\lambda,\beta)).$$

As shown previously,

$$J((\Pi_{221},\theta_2,\lambda,\beta),(\Pi_{22},\lambda,\beta))|_{\lambda=0} = J((\Pi_{221},\theta_2,\lambda,\beta),(\Pi_{22},\lambda,\beta)).$$

It also holds that

$$J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta)) = J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta))|_{\lambda=0} + W,$$

where

$$W = \left(0 \quad (B'_\perp\lambda' \otimes I_{k_2})\frac{\partial\,\text{vec}(\theta_\perp)}{\partial\,\text{vec}(\theta_2)'} \quad (I_m \otimes \theta_\perp\lambda)\frac{\partial\,\text{vec}(B_\perp)}{\partial\,\text{vec}(\beta)'} \quad 0\right)$$

such that

$$(J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta))|_{\lambda=0})W' = 0.$$

This implies that

$$\begin{aligned}
|J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta))| &= |J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta))J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta))'|^{1/2} \\
&= |(J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta))|_{\lambda=0})(J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta))|_{\lambda=0})' + WW'|^{1/2} \\
&\geq |(J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta))|_{\lambda=0})(J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta))|_{\lambda=0})'|^{1/2} \\
&\geq |(J(\Phi,(\Pi_{221},\theta_2,\lambda,\beta))|_{\lambda=0})|
\end{aligned}$$

and consequently

$$|J(\Phi,(\Pi_{22},\lambda,\beta))| \geq |(J(\Phi,(\Pi_{22},\lambda,\beta))|_{\lambda=0})|.$$

APPENDIX B

Proof of Theorem 1. Assume that the reduced form of the SEM,

$$Y_{\bar{m}} B_{\bar{m}\bar{m}} = Z_{\bar{m}} \Gamma_{\bar{m}\bar{m}} + Z_{\bar{m}m} \Gamma_{m\bar{m}} + \varepsilon_{\bar{m}},$$

reads

$$Y_{\bar{m}} = Z_{\bar{m}} \Pi_{\bar{m}\bar{m}} + Z_{\bar{m}m} \Pi_{m\bar{m}} + \xi_{\bar{m}},$$

where $\Pi_{\bar{m}\bar{m}} = \Gamma_{\bar{m}\bar{m}} B_{\bar{m}\bar{m}}^{-1}$, $\Pi_{m\bar{m}} = \Gamma_{m\bar{m}} B_{\bar{m}\bar{m}}^{-1}$, and this reduced form is equivalent to a set of nonlinear (reduced rank) restrictions on the parameters of a linear model and the (hyper) parameters of this linear model, which are restricted to zero to obtain the reduced form, are locally uncorrelated with specific other parameters.

The parameter matrix of the reduced form of the SEM from Theorem 1 reads

$$\begin{pmatrix} \Gamma_{\bar{m}\bar{m}} & 0 \\ \Gamma_{m\bar{m}} & \Gamma_{\bar{m}m} \\ 0 & \Gamma_{mm} \end{pmatrix} \begin{pmatrix} B_{\bar{m}\bar{m}} & B_{\bar{m}m} \\ B_{m\bar{m}} & B_{mm} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \Gamma_{\bar{m}\bar{m}} & 0 \\ \Gamma_{m\bar{m}} & \Gamma_{\bar{m}m} \\ 0 & \Gamma_{mm} \end{pmatrix} \begin{pmatrix} B_{\bar{m}\bar{m}}^{-1} + B_{\bar{m}\bar{m}}^{-1} B_{\bar{m}m} B_{mm\cdot\bar{m}}^{-1} B_{m\bar{m}} B_{\bar{m}\bar{m}}^{-1} & -B_{\bar{m}\bar{m}}^{-1} B_{\bar{m}m} B_{mm\cdot\bar{m}}^{-1} \\ -B_{mm\cdot\bar{m}}^{-1} B_{m\bar{m}} B_{\bar{m}\bar{m}}^{-1} & B_{mm\cdot\bar{m}}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \Pi_{\bar{m}\bar{m}}(I_{j_{\bar{m}}} + \beta_{\bar{m}m}\beta_{m\bar{m}}) & -\Pi_{\bar{m}\bar{m}}\beta_{\bar{m}m} \\ \Pi_{m\bar{m}}(I_{j_{\bar{m}}} + \beta_{\bar{m}m}\beta_{m\bar{m}}) - \Pi_{\bar{m}m}\beta_{m\bar{m}} & \Pi_{\bar{m}m} - \Pi_{m\bar{m}}\beta_{\bar{m}m} \\ -\Pi_{mm}\beta_{m\bar{m}} & \Pi_{mm} \end{pmatrix},$$

where $\Pi_{\bar{m}\bar{m}} = \Gamma_{\bar{m}\bar{m}} B_{\bar{m}\bar{m}}^{-1}$, $\Pi_{m\bar{m}} = \Gamma_{m\bar{m}} B_{\bar{m}\bar{m}}^{-1}$, $\Pi_{mm} = \Gamma_{mm} B_{mm\cdot\bar{m}}^{-1}$, $\Pi_{\bar{m}m} = \Gamma_{\bar{m}m} B_{mm\cdot\bar{m}}^{-1}$, $B_{\bar{m}\bar{m}\cdot m} = B_{\bar{m}\bar{m}} - B_{\bar{m}m} B_{mm}^{-1} B_{m\bar{m}} = B_{\bar{m}\bar{m}} - B_{\bar{m}m} B_{m\bar{m}}$, $B_{mm\cdot\bar{m}} = B_{mm} - B_{m\bar{m}} B_{\bar{m}\bar{m}}^{-1} B_{\bar{m}m}$, $\beta_{m\bar{m}} = B_{m\bar{m}} B_{\bar{m}\bar{m}}^{-1}$, $\beta_{\bar{m}m} = B_{\bar{m}m} B_{mm\cdot\bar{m}}^{-1}$. This implies, as both Γ_{mm} and $\Gamma_{\bar{m}m}$ are unrestricted, that no restrictions are imposed on Π_{mm} and $\Pi_{\bar{m}m}$. The linear model of which the reduced form is a nonlinear restriction reads

$$(Y_{\bar{m}} \quad Y_m) = (Z_{\bar{m}} \quad Z_{\bar{m}m} \quad Z_m)\Phi + \xi,$$

where $\Phi: (l_{\bar{m}} + l_m + i_m) \times (j_{\bar{m}} + j_{\bar{m}})$ and can be specified as

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \\ \Phi_{31} & \Phi_{23} \end{pmatrix},$$

where $\Phi_{11}: l_{\bar{m}} \times j_{\bar{m}}$, $\Phi_{21}: l_m \times j_{\bar{m}}$, $\Phi_{31}: i_m \times j_{\bar{m}}$, $\Phi_{12}: l_{\bar{m}} \times j_m$, $\Phi_{22}: l_m \times j_m$, $\Phi_{23}: i_m \times j_m$. To obtain the restrictions on the linear model parameters that result in the reduced form, we specify Φ as

$$\begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \\ \Phi_{31} & \Phi_{23} \end{pmatrix} = \begin{pmatrix} \Theta_{11} \\ \Theta_{21} \\ 0 \end{pmatrix} (I_{j_{\bar{m}}} \quad 0) + \begin{pmatrix} \Theta_{12} \\ \Theta_{22} \\ \Pi_{mm} \end{pmatrix} (-\beta_{m\bar{m}} \quad I_{j_m})$$

$$+ \begin{pmatrix} 0 \\ 0 \\ \Pi_{mm\perp} \lambda_{mm} (-\beta_{m\bar{m}} \quad I_{j_m})_\perp \end{pmatrix},$$

$$\begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} = \begin{pmatrix} \Pi_{\bar{m}\bar{m}} \\ \Pi_{m\bar{m}} \end{pmatrix} (I_{j_{\bar{m}}} \quad -\beta_{\bar{m}m}) + \begin{pmatrix} 0 \\ I_{j_m} \end{pmatrix} \Pi_{\bar{m}m} (0 \quad I_{j_m})$$

$$+ \begin{pmatrix} \Pi_{\bar{m}\bar{m}\perp} \lambda_{\bar{m}\bar{m}} (I_{j_{\bar{m}}} \quad -\beta_{\bar{m}m})_\perp \\ 0 \end{pmatrix},$$

where $\Theta_{11}: l_{\bar{m}} \times j_{\bar{m}}$, $\Theta_{21}: l_m \times j_{\bar{m}}$, $\Theta_{12}: l_{\bar{m}} \times j_m$, $\Theta_{22}: l_m \times j_m$. It is clear from the chosen specification that when $\lambda_{mm} = 0$, $\lambda_{\bar{m}\bar{m}} = 0$, the reduced form results and that λ_{mm} is locally uncorrelated (when it is zero) with the parameters contained in Π_{mm} and $\beta_{m\bar{m}}$, and $\lambda_{\bar{m}\bar{m}}$ is locally uncorrelated (when it is zero) with the parameters contained in $\Pi_{\bar{m}\bar{m}}$ and $\beta_{\bar{m}m}$. As we can apply the same kind of decomposition on Π_{mm} and $\Pi_{m\bar{m}}$, which we assumed to be possible, and because $\Pi_{\bar{m}m}$ and Π_{mm} are unrestricted, such that there is no need to decompose them further, we can recursively apply the preceding decomposition and thereby the theorem is proved. ■

APPENDIX C. SINGULAR VALUE DECOMPOSITION AND JACOBIANS TWO EQUATION MODEL

For the two equation model, reduced rank restrictions are imposed on the parameter matrices Φ_2 and Φ_3. In the following we state the SVD's and the Jacobians involved with these two parameter matrices. We start with Φ_2.

$$\Phi_2 = (\psi \quad \psi_\perp) \begin{pmatrix} \Pi_{211} & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} B_2 \\ B_{2\perp} \end{pmatrix}$$

$$= \psi \Pi_{211} B_2 + \psi_\perp \lambda_2 B_{2\perp},$$

where $\Pi_{21} = (\Pi'_{211}\ \Pi'_{212})'$, $\Pi_{211}: m_1 \times m_1$, $\Pi_{212}: (k_2 - m_1) \times m_1$, $\psi_2 = \Pi_{212}\Pi_{211}^{-1}$, $\psi = (I_{m_1}\ \psi_2')'$, $\psi_\perp = (-\psi_2\ I_{k_2-m_1})'(I_{k_2-m_1} + \psi_2\psi_2')^{-1/2}$, $B_2 = (I_{m_1}\ \beta_2)$, $B_{2\perp} = (I_{m_2} + \beta_2'\beta_2)^{-1/2}(-\beta_2'\ I_{m_2})$. A SVD can be used to obtain these parameters from Φ_2,

$$\Phi_2 = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}',$$

where $U'U = I_{k_2}$; $V'V = I_m$; $U_{11}, S_1, V_{11}: m_1 \times m_1$; $S_2: (k_2 - m_1) \times m_2$; $V_{22}: m_2 \times m_2$; $U_{21}: (k_2 - m_1) \times m_1$; $U_{12}: m_1 \times (k_2 - m_1)$; $U_{22}: (k_2 - m_1) \times (k_2 - m_1)$; $V_{21}, V_{12}': m_2 \times m_1$; and S_2 contains the smallest m_2 singular values of Φ_2. This leads to the relations

$$\Pi_{211} = U_{11}S_1V_{11}', \qquad \psi_2 = U_{21}U_{11}^{-1},$$
$$\beta_2 = (V_{21}V_{11}^{-1})', \qquad \lambda_2 = (U_{22}U_{22}')^{-1/2}U_{22}S_2V_{22}'(V_{22}V_{22}')^{-1/2}.$$

The Jacobians of Φ_2 with respect to Π_{211}, ψ_2, β_2, and λ_2 read

$$J_1 = \frac{\partial \operatorname{vec}(\Phi_2)}{\partial \operatorname{vec}(\Pi_{221})'} = (B_2' \otimes \psi),$$

$$J_2 = \frac{\partial \operatorname{vec}(\Phi_2)}{\partial \operatorname{vec}(\psi_2)'} = (B_2'\Pi_{211}' \otimes I_{k_2})\frac{\partial \operatorname{vec}(\psi)}{\partial \operatorname{vec}(\psi_2)'} + (B_{2\perp}'\lambda_2' \otimes I_{k_2})\frac{\partial \operatorname{vec}(\psi_\perp)}{\partial \operatorname{vec}(\psi_2)'},$$

$$J_3 = \frac{\partial \operatorname{vec}(\Phi_2)}{\partial \operatorname{vec}(\beta_2)'} = (I_m \otimes \psi\Pi_{211})\frac{\partial \operatorname{vec}(B_2)}{\partial \operatorname{vec}(\beta_2)'} + (I_m \otimes \psi_\perp\lambda_2)\frac{\partial \operatorname{vec}(B_{2\perp})}{\partial \operatorname{vec}(\beta_2)'},$$

$$J_4 = \frac{\partial \operatorname{vec}(\Phi_2)}{\partial \operatorname{vec}(\lambda_2)'} = (B_{2\perp}' \otimes \psi_\perp),$$

where

$$\frac{\partial \operatorname{vec}(\psi)}{\partial \operatorname{vec}(\psi_2)'} = \left(I_{m_1} \otimes \begin{pmatrix} 0 \\ I_{k_2-m_1} \end{pmatrix}\right),$$

$$\frac{\partial \operatorname{vec}(\psi_\perp)}{\partial \operatorname{vec}(\psi_2)'} = -\left(H^{-(1/2)'} \otimes \begin{pmatrix} I_{m_1} \\ 0 \end{pmatrix}\right)K_{k_2-m_1,m_1}$$
$$+ \left(I_{k_2-m_1} \otimes \begin{pmatrix} -\psi_2' \\ I_{k_2-m_1} \end{pmatrix}\right)\frac{\partial \operatorname{vec}(H^{-1/2})}{\partial \operatorname{vec}(H^{1/2})'}\frac{\partial \operatorname{vec}(H^{1/2})}{\partial \operatorname{vec}(H)'}\frac{\partial \operatorname{vec}(H)}{\partial \operatorname{vec}(\psi_2)'},$$

$$\frac{\partial \operatorname{vec}(H^{-1/2})}{\partial \operatorname{vec}(H^{1/2})'} = -(H^{-(1/2)'} \otimes H^{-1/2}),$$

$$\frac{\partial \operatorname{vec}(H^{1/2})}{\partial \operatorname{vec}(H)'} = ((I_{k_2-m_1} \otimes H^{1/2}) + (H^{(1/2)'} \otimes I_{k_2-m_1}))^{-1},$$

$$\frac{\partial \operatorname{vec}(H)}{\partial \operatorname{vec}(\psi_2)'} = (\psi_2 \otimes I_{k_2-m_1}) + (I_{k_2-m_1} \otimes \psi_2)K_{k_2-m_1,m_1},$$

$$\frac{\partial\,\mathrm{vec}(B_2)}{\partial\,\mathrm{vec}(\beta_2)'} = \left(\begin{pmatrix} 0 \\ I_{m_2} \end{pmatrix} \otimes I_{m_1}\right),$$

$$\frac{\partial\,\mathrm{vec}(B_{2\perp})}{\partial\,\mathrm{vec}(\beta_2)'} = -\left(\begin{pmatrix} I_{m_1} \\ 0 \end{pmatrix} \otimes \mathcal{B}^{-1/2}\right) K_{m_1,m_2}$$
$$+ ((-\beta_2' \quad I_{m_2})' \otimes I_{m_2}) \frac{\partial\,\mathrm{vec}(\mathcal{B}^{-1/2})}{\partial\,\mathrm{vec}(\mathcal{B}^{1/2})'} \frac{\partial\,\mathrm{vec}(\mathcal{B}^{1/2})}{\partial\,\mathrm{vec}(\mathcal{B})'} \frac{\partial\,\mathrm{vec}(\mathcal{B})}{\partial\,\mathrm{vec}(\beta)'},$$

$$\frac{\partial\,\mathrm{vec}(\mathcal{B}^{-1/2})}{\partial\,\mathrm{vec}(\mathcal{B}^{1/2})'} = -(\mathcal{B}^{-(1/2)'} \otimes \mathcal{B}^{-1/2}),$$

$$\frac{\partial\,\mathrm{vec}(\mathcal{B}^{1/2})}{\partial\,\mathrm{vec}(\mathcal{B})'} = ((I_{m_2} \otimes \mathcal{B}^{1/2}) + (\mathcal{B}^{(1/2)'} \otimes I_{m_2}))^{-1},$$

$$\frac{\partial\,\mathrm{vec}(\mathcal{B}^{1/2})}{\partial\,\mathrm{vec}(\beta_2)'} = (\beta_2' \otimes I_{m_2})K_{m_1,m_2} + (I_{m_2} \otimes \beta_2')$$

and $H = I_{k_2-m_1} + \psi_2\psi_2'$, $H^{1/2}H^{1/2} = H$, $\mathcal{B} = (I_{m_2} + \beta_2'\beta_2)$, $\mathcal{B}^{1/2}\mathcal{B}^{1/2} = \mathcal{B}$. The Jacobian of the transformation from Φ_2 to $(\Pi_{211},\psi_2,\beta_2,\lambda_2)$ then reads

$$\frac{\partial\,\mathrm{vec}(\Phi_2)}{\partial(\mathrm{vec}(\Pi_{211})'\ \mathrm{vec}(\psi_2)'\ \mathrm{vec}(\beta_2)'\ \mathrm{vec}(\lambda_2)')} = (J_1 \quad J_2 \quad J_3 \quad J_4).$$

Because $\psi_2 = \Pi_{212}\Pi_{211}^{-1}$, the Jacobians of the transformations from $(\Pi_{211},\psi_2,\beta_2,\lambda_2)$ to Π_{211}, Π_{212}, β_2, and λ_2 read

$$G_1 = \frac{\partial(\mathrm{vec}(\Pi_{211})'\mathrm{vec}(\psi_2)'\mathrm{vec}(\beta_2)'\ \mathrm{vec}(\lambda_2)')'}{\partial\,\mathrm{vec}(\Pi_{211})'} = \begin{pmatrix} I_{m_1} \otimes I_{m_1} \\ -\Pi_{211}^{-1\prime} \otimes \Pi_{212}\Pi_{211}^{-1} \\ 0 \\ 0 \end{pmatrix},$$

$$G_2 = \frac{\partial(\mathrm{vec}(\Pi_{211})'\ \mathrm{vec}(\psi_2)'\ \mathrm{vec}(\beta_2)'\ \mathrm{vec}(\lambda_2)')'}{\partial\,\mathrm{vec}(\Pi_{212})'} = \begin{pmatrix} 0 \\ \Pi_{211}^{-1\prime} \otimes I_{k_2-m_1} \\ 0 \\ 0 \end{pmatrix},$$

$$G_3 = \frac{\partial(\mathrm{vec}(\Pi_{211})'\mathrm{vec}(\psi_2)'\ \mathrm{vec}(\beta_2)'\ \mathrm{vec}(\lambda_2)')'}{\partial\,\mathrm{vec}(\beta_2)'} = \begin{pmatrix} 0 \\ 0 \\ I_{m_2} \otimes I_{m_1} \\ 0 \end{pmatrix},$$

$$G_4 = \frac{\partial(\mathrm{vec}(\Pi_{211})'\ \mathrm{vec}(\psi_2)'\ \mathrm{vec}(\beta_2)'\ \mathrm{vec}(\lambda_2)')'}{\partial\,\mathrm{vec}(\lambda_2)'} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ I_{m_2} \otimes I_{k_2-m_1} \end{pmatrix}.$$

The Jacobian of the transformation from Φ_2 to $(\Pi_{21}, \beta_2, \lambda_2)$ then becomes

$$
\begin{aligned}
|J(\Phi_2,(\Pi_{21},\beta_2,\lambda_2))| &= \left| \frac{\partial \operatorname{vec}(\Phi_2)}{\partial(\operatorname{vec}(\Pi_{21})' \operatorname{vec}(\beta_2)' \operatorname{vec}(\lambda_2)')} \right| \\
&= \left| \frac{\partial \operatorname{vec}(\Phi_2)}{\partial(\operatorname{vec}(\Pi_{211})' \operatorname{vec}(\psi_2)' \operatorname{vec}(\beta_2)' \operatorname{vec}(\lambda_2)')} \right| \\
&\quad \times \left| \frac{\partial(\operatorname{vec}(\Pi_{211})' \operatorname{vec}(\psi_2)' \operatorname{vec}(\beta_2)' \operatorname{vec}(\lambda_2)')'}{\partial(\operatorname{vec}(\Pi_{21})' \operatorname{vec}(\beta_2)' \operatorname{vec}(\lambda_2)')} \right| \\
&= |(J_1 \quad J_2 \quad J_3 \quad J_4)||(G_1 \quad G_2 \quad G_3 \quad G_4)|.
\end{aligned}
$$

The specification of Φ_3 reads

$$
\Phi_3 = (\theta \quad \theta_\perp) \begin{pmatrix} \Pi_{321} & 0 \\ 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} B_1 \\ B_{1\perp} \end{pmatrix},
$$

where $\theta = (I_{m_2} \;\; \theta_2')'$, $B_1 = (\beta_1 \;\; I_{m_2})$, $\Pi_{32} = (\Pi_{321}' \;\; \Pi_{322}')'$, $\Pi_{321}: m_2 \times m_2$, $\Pi_{322}: (k_3 - m_2) \times m_2$, $\theta_2 = \Pi_{322}\Pi_{321}^{-1}$. So, the specification of Φ_3 is identical to the specification of Φ for the INSEM. The parameters $(\Pi_{32}, \beta_1, \lambda_3)$ can therefore be obtained using the SVD's (3.3)–(3.5) and changing the sizes of the involved matrices, i.e., k_2 to k_3, $m-1$ to m_2, 1 to m_1. Also the Jacobian involved in the parameter transformation of the INSEM is identical to the Jacobian in the case of Φ_3 when we change the sizes of the involved matrices in the outlined manner.

APPENDIX D. SINGULAR VALUE DECOMPOSITION AND JACOBIANS THREE EQUATION MODEL

For the three equation model, reduced rank restrictions are imposed on the parameter matrices $(\Phi_2' \;\; \Phi_3')'$, Θ, and Φ_1. The important difference with the INSEM and the two equation model lies in Θ, which itself already results from a reduced rank restriction. As we have to analyze Θ given $(\Phi_2' \;\; \Phi_3')'$, we start with the SVD and Jacobian involved with $(\Phi_2' \;\; \Phi_3')'$. The specification of $(\Phi_2' \;\; \Phi_3')'$ reads

$$
\begin{pmatrix} \Phi_2 \\ \Phi_3 \end{pmatrix} = \Theta \begin{pmatrix} \beta_{21} & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{pmatrix} + \Theta_\perp \lambda_2 \begin{pmatrix} \beta_{21} & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{pmatrix}_\perp.
$$

This implies that when $\Phi_2 = (\Phi_{21} \;\; \Phi_{22})$, $\Phi_{21}: k_2 \times (m_1 + m_2)$, $\Phi_{22}: k_2 \times m_3$; $\Phi_3 = (\Phi_{31} \;\; \Phi_{32})$, $\Phi_{31}: k_3 \times (m_1 + m_2)$, $\Phi_{32}: k_3 \times m_3$;

$$
\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix},
$$

$\Theta_{11}: k_2 \times (m_1 + m_2)$, $\Theta_{12}: k_2 \times m_3$, $\Theta_{21}: k_3 \times (m_1 + m_2)$, $\Theta_{22}: k_3 \times m_3$ that the following equality holds:

$$\begin{pmatrix} \Theta_{12} \\ \Theta_{22} \end{pmatrix} = \begin{pmatrix} \Phi_{22} \\ \Phi_{32} \end{pmatrix}$$

and we are left with

$$\begin{pmatrix} \Phi_{21} \\ \Phi_{31} \end{pmatrix} = \begin{pmatrix} \Theta_{11} \\ \Theta_{21} \end{pmatrix} (\beta_{21} \quad I_{m_2}) + \begin{pmatrix} \Theta_{11} \\ \Theta_{21} \end{pmatrix}_{\perp} \lambda_2 (\beta_{21} \quad I_{m_2})_{\perp},$$

which is again identical to the specification of Φ for the INSEM such that when we change the sizes of the matrices in the appropriate manner, i.e., k_2 to $k_2 + k_3$, $m-1$ to m_2, and 1 to m_3, we can directly use the SVD's and Jacobians for Φ of the INSEM.

The SVD's and Jacobians for Θ_2 are constructed using (6.17) and (6.18),

$$(\Theta_{21} \quad \Theta_{22}) = \Pi_{33}(\beta_{32} \quad I_{m_3}) + \Pi_{33\perp} \lambda_3 (\beta_{32} \quad I_{m_3})_{\perp}.$$

Again this specification is identical to the specification of Φ for the INSEM such that we can use the SVD and Jacobians specified for the INSEM when we change the sizes of the matrices in the appropriate manner, i.e., k_2 to k_3, $m-1$ to m_3, and 1 to m_2.

The specification of Φ_1 reads

$$\Phi_1 = (\Pi_{11} \quad \Pi_{12}) \left(I_{m_1+m_2} \quad \begin{pmatrix} \beta_{13} \\ \beta_{23} \end{pmatrix} \right) + (\Pi_{11} \quad \Pi_{12})_{\perp} \lambda_1 \left(I_{m_1+m_2} \quad \begin{pmatrix} \beta_{13} \\ \beta_{23} \end{pmatrix} \right)_{\perp}.$$

This specification is identical to the specification of Φ_2 in the two equation model such that we can use the Jacobians and the SVD listed there when we change the sizes of the matrices in the appropriate manner, i.e., k_2 to k_1, m_1 to $m_1 + m_2$, and m_2 to m_3.

APPENDIX E. OBTAINING SEM's FROM LINEAR MODELS USING SVD's

The specification of Φ reads

$$\Phi = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} v'_{11} & V'_{21} \\ v'_{12} & v'_{22} \end{pmatrix}$$

$$= \Pi_{22}(\beta \quad I_{m-1}) + \Pi_{22\perp} \lambda (\beta \quad I_{m-1})_{\perp},$$

such that

$$\Pi_{22}(\beta \quad I_{m-1}) = \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} S_1 (v'_{11} \quad V'_{21})$$

and

$$\Pi_{22\perp}\lambda(\beta \quad I_{m-1})_\perp = \begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix} s_2(v_{12}' \quad v_{22}').$$

Consequently,

$$\Pi_{22} = \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} S_1 V_{21}' \quad \text{and} \quad \beta = V_{21}'^{-1} v_{11}'.$$

Substituting these expressions in the specification of $\Pi_{22\perp}$ and $(\beta \quad I_{m-1})_\perp$ gives

$$\begin{aligned}
\Pi_{22\perp} &= \begin{pmatrix} -\Pi_{221}^{-1\prime}\Pi_{222}' \\ I_{k_2-m+1} \end{pmatrix} \left(I_{k_2-m+1} + \Pi_{222}\Pi_{221}^{-1}\Pi_{221}^{-1\prime}\Pi_{222}'\right)^{-1/2} \\
&= \begin{pmatrix} -U_{11}'^{-1}S_1^{-1}V_{21}^{-1}V_{21}S_1U_{21}' \\ I_{k_2-m+1} \end{pmatrix} \left(I_{k_2-m+1} + U_{21}U_{11}^{-1}U_{11}'^{-1}U_{21}'\right)^{-1/2} \\
&= \begin{pmatrix} -U_{11}'^{-1}U_{21}' \\ I_{k_2-m+1} \end{pmatrix} \left(I_{k_2-m+1} + U_{21}U_{11}^{-1}U_{11}'^{-1}U_{21}'\right)^{-1/2},
\end{aligned}$$

as $U_{11}'U_{12} + U_{21}'U_{22} = 0$ (because of the orthogonality of U), $U_{12}U_{22}^{-1} = -U_{11}'^{-1}U_{21}'$, and $U_{12}'U_{12} + U_{22}'U_{22} = I_{k_2-m+1}$, such that

$$\begin{aligned}
\Pi_{22\perp} &= \begin{pmatrix} U_{12}U_{22}^{-1} \\ I_{k_2-m+1} \end{pmatrix} \left(I_{k_2-m+1} + U_{22}^{-1\prime}U_{12}'U_{12}U_{22}^{-1}\right)^{-1/2} \\
&= \begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix} U_{22}^{-1}(U_{22}'^{-1}(U_{22}'U_{22} + U_{12}'U_{12})U_{22}^{-1})^{-1/2} \\
&= \begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix} U_{22}^{-1}((U_{22}'^{-1}U_{22}^{-1})^{-1})^{1/2} \\
&= \begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix} U_{22}^{-1}(U_{22}U_{22}')^{1/2}.
\end{aligned}$$

Similarly for $(\beta \quad I_{m-1})_\perp$,

$$\begin{aligned}
(\beta \quad I_{m-1})_\perp &= (1 + \beta'\beta)^{-1/2}(1 \quad -\beta') \\
&= (1 + v_{11}V_{21}^{-1}V_{21}'^{-1}v_{11}')^{-1/2}(1 \quad -v_{11}V_{21}^{-1}) \\
&= (1 + v_{12}^{-1\prime}v_{22}'v_{22}v_{12}^{-1})^{-1/2}(1 \quad v_{12}^{-1\prime}v_{22}') \\
&= (v_{12}^{-1\prime}(v_{12}'v_{12} + v_{22}'v_{22})v_{12}^{-1})^{-1/2}v_{12}'^{-1}(v_{12}' \quad v_{22}') \\
&= (v_{12}^{-1\prime}v_{12}^{-1})^{-1/2}v_{12}'^{-1}(v_{12}' \quad v_{22}') \\
&= (v_{12}v_{12}')^{1/2}v_{12}'^{-1}(v_{12}' \quad v_{22}'),
\end{aligned}$$

because $v_{11}'v_{12} + V_{21}'v_{22} = 0$, such that $-V_{21}'^{-1}v_{11}' = v_{22}v_{12}^{-1}$, and $v_{12}'v_{12} + v_{22}'v_{22} = 1$. Consequently, to have equivalence,

$$\begin{aligned}\lambda &= (U_{22}U_{22}')^{-1/2}U_{22}s_2v_{12}'(v_{12}v_{12}')^{-1/2}\\ &= bs_2a,\end{aligned}$$

where $b = (U_{22}U_{22}')^{-1/2}U_{22}$, and $a = v_{12}'(v_{12}v_{12}')^{-1/2}$. Both b and a are orthogonal matrices (scalars) that result from the singular value decomposition because when $X = USV'$, where both U and V are orthogonal, then

$$(XX')^{1/2} = (USV'VSU')^{1/2} = (US^2U')^{1/2} = USU',$$

such that

$$(XX')^{-1/2}X = US^{-1}U'USV' = UV',$$

which is an orthogonal matrix, such that λ equals the smallest singular value pre- and postmultiplied by orthogonal vectors/matrices, or stated differently, λ is a rotation of the singular values.