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Methods

Strategic Pricing in Volatile Markets

Sebastian Gryglewicz,[a] Aaron Kolb[b,*]

[a] Erasmus School of Economics, Erasmus University Rotterdam, Rotterdam 3000DR, Netherlands; [b] Kelley School of Business, Indiana University, Bloomington, Indiana 47405

*Corresponding author

Contact: gryglewicz@ese.eur.nl, https://orcid.org/0000-0002-6474-0185 (SG); kolba@indiana.edu, https://orcid.org/0000-0002-1467-6031 (AK)

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Abstract. We study dynamic entry deterrence through limit pricing in markets subject to persistent demand shocks. An incumbent is privately informed about its costs, high or low, and can deter a Bayesian potential entrant by setting its prices strategically. The entrant can irreversibly enter the market at any time for a fixed cost, earning a payoff that depends on the market conditions and the incumbent's unobserved type. Market demand evolves as a geometric Brownian motion. When market demand is low, entry becomes a distant threat, so there is little benefit to further deterrence, and, in equilibrium, a weak incumbent becomes tempted to reveal itself by raising its prices. We characterize a unique equilibrium in which the entrant enters when market demand is sufficiently high (relative to the incumbent's current reputation), and the weak incumbent mixes over revealing itself when market demand is sufficiently low. In this equilibrium, pricing and entry decisions exhibit path dependence, depending not only on the market's current size, but also its historical minimum.

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1. Introduction

Firms use pricing strategically to influence entry and exit of competitors. A key form of strategic pricing is limit pricing, whereby a market incumbent sets low prices to deter a potential entrant. Some of the earliest formal analyses on this topic include Gaskins (1971) and Deshmukh and Winston (1979), whereas Milgrom and Roberts (1982) introduced a signaling explanation for limit pricing. The airline industry provides striking examples of this phenomenon, and several papers have documented evidence of lower ticket prices when there is more potential competition (Morrison and Winston 1987, Morrison 2001, Kwoka and Shumilkina 2010). Goolsbee and Syverson (2008) and Sweeting et al. (2020) study the "Southwest Effect" using data from the 1990 s and 2000 s and show that ticket prices decline on routes when Southwest begins serving routes out of both endpoints of the route in question, thereby becoming a potential entrant.

In practice, strategic pricing takes place in dynamic, stochastic environments: An incumbent that uses prices to signal unprofitable entry, in reality, has to do so repeatedly and under changing market conditions. As strategic pricing is, in essence, a trade-off between short-term costs and long-term benefits, incentives to engage in it depend on market demand, which evolves over time. Similarly, an entrant's decision to enter depends on changing market conditions. The purpose of this paper is to analyze the dynamics of these interactions in a setting where market demand is subject to persistent shocks.

We study a two-player, continuous-time model, in which a market incumbent is privately informed about its marginal costs, which determine its type, strong or weak. The potential entrant is uncertain about the incumbent's marginal costs and can enter the market at a fixed cost; the entrant then earns a positive continuation payoff that is increasing in market demand if the incumbent is weak, but nothing if the incumbent is strong. The incumbent of the weak type gets a reduced payoff after entry, but can deter entry through a costly signaling action that can be interpreted as setting low prices to mimic a strong incumbent. We model market demand as a geometric Brownian motion X, so shocks have a persistent effect. We allow this process to have upward or downward drift, and, therefore, our model applies equally to markets that are increasing or declining, on average.

How does the incumbent signal in such a dynamic setting? How does the entrant strategically time its entry? Our primary insight is that when the market is subject to persistent shocks, the incentive to signal disappears in some market conditions. The entrant wants to enter when demand becomes high enough relative to its updated belief that the incumbent is strong. On the other hand, if demand gets low enough, there is little threat of entry in the near future, so the weak type of incumbent no longer wants to send costly signals and, consequently, becomes willing to reveal its type by raising prices.

We characterize a unique equilibrium within the following subclass of Markov equilibria: The weak incumbent reveals with some probability whenever the process X falls below some decreasing function L of the current belief, and the entrant enters the market at an upper boundary U that is increasing the current belief. By continuing to signal at low demand, the incumbent builds its reputation, which delays entry further and makes it more willing to continue signaling. The state variable in this equilibrium is two-dimensional, consisting of current market demand and the incumbent's current reputation.

The decisions of the two players are strategically interrelated. The equilibrium strategy U of the entrant balances the benefit of profitably participating in the market, given the current belief about the incumbent, and twofold opportunity costs: (i) Demand may soon rise, creating a more favorable entry opportunity; or (ii) if demand falls, the weak type of incumbent may reveal itself. The strategy L of the weak type of incumbent trades off, on one hand, the value of deterring entry until the market size is U by continuing to signal and, on the other hand, the value of increased flow payoffs from ceasing to signal, but revealing its type and thus facing earlier entry by a now-emboldened entrant.

When the incumbent is believed to play according to a threshold L, the incumbent's reputation, conditional on continuing to signal, is determined by (the prior and) the historical minimum process $M_t = \min_{0\leq s\leq t} X_s$. Hence, for any threshold L of the incumbent, the entrant's best response problem can be cast as one of optimal stopping of the process (X, M). A key difference relative to the classic analysis of Peskir (1998) is that our entrant's payoff as a function of X and M is mediated by L, which, in turn, depends on U because it must be a best reply. Thus, the thresholds (U, L) must be determined jointly, and, a priori, there is a possibility of multiple equilibria.

To establish uniqueness of the (U, L) equilibrium, we draw from the theory of differential equations the concept of an *antifunnel* (Hubbard and West 1991). We first argue that an upper bound on the entrant's entry threshold function is the best response given a "best-case scenario," where the incumbent fully separates at a particular lower threshold. Similarly, a lower bound corresponds to a "worst-case scenario," where the incumbent pools forever (and, thus, no information is revealed about its type). We show that these upper and lower bounds form an antifunnel for the differential equation for the entry threshold in the following sense: Any solution to this differential equation that escapes through the upper (respectively (resp.), lower) bound must forever stay above (resp., below) it. We then show that these bounds converge sufficiently fast, such that there is a unique solution to the differential equation that lies between them. Some intuition is as follows. If the entrant were to raise the entry threshold, there would be more at stake for the weak type of incumbent in maintaining a reputation. The incumbent's threshold would thus decrease. But this implies that the entrant must wait longer to obtain information, which reduces the option value of waiting, and, in response, the entrant should decrease the entry threshold.

We use equilibrium price and reputation dynamics to derive several implications. The stochastic limit-pricing game implies that price dynamics may reveal limit pricing of incumbents. Specifically, in equilibrium, the limit-pricing weak incumbent reveals its type by *increasing* prices as the market conditions become unfavorable to entry and the incumbent's position becomes relatively secure. In other words, our results offer an explanation for why prices may rise in falling markets. Moreover, to an external observer, this phenomenon may serve as an indicator of entry-deterring limit pricing.

We also show that the decision of the entrant to enter exhibits path dependence—although demand is modeled as a Markovian variable, the entrant's assessment of entry profitability depends on both current demand and its historical minimum, as the latter determines the incumbent's reputation. Jaske and Watkins (2020) provide empirical evidence from the airline industry for a related form of path dependence, with a lower running minimum of market demand being correlated with higher prices by incumbents. This is consistent with our model because our incumbent is more likely to have ceased limit pricing after periods of low demand.

Our model also implies that the learning mechanism postpones entry. In a dynamic stochastic environment, the well-known option value of waiting (McDonald and Siegel 1986) delays entry decisions: The entrant waits to observe future demand realizations. However, in our setup, the entrant delays entry even further, in anticipation of possible learning about the incumbent's type.

Our paper contributes to the existing literature on investment and other irreversible decisions under uncertainty. Early papers in this literature include Dixit (1989) and McDonald and Siegel (1986). More recently, Kwon (2010) studies both investment and exit decisions in a single-player model, where demand, like in our setting, evolves according to a Brownian motion. Sunar et al. (2021) study a sequential duopoly problem, where firms choose both the size and timing of their investment in a market with unknown favorableness, and the leader's observed (Brownian) earnings provide information about the market's conditions. Our innovation relative to this literature is that we feature a privately informed rival player (namely, our incumbent), whose strategic behavior itself transmits information. This induces an additional stochastic process (the Bayesian posterior belief about her type, or, equivalently, the historical minimum of market demand) that factors into the entrant's optimal stopping problem beyond the current market demand alone.[1] Methodologically, then, the entrant's problem in our model relates to work on single-player, multidimensional optimization problems, such as the optimal stopping of the maximum process (Peskir 1998), financial lookback options (Guo and Shepp 2001), and investment timing, where the value of the investment has unknown drift (Décamps et al. 2005).

A number of other papers study dynamic aspects of signaling. Saloner (1984) presents a multiperiod version of the limit-pricing model of Matthews and Mirman (1983), in which signals received by the uninformed firm are noisy. Heinsalu (2018) and Dilmé (2019a) study dynamic signaling in a more abstract setup with noisy observation of signaling efforts. In contrast, we assume that the actions of the incumbent are observed directly by the entrant. The key difference in our model is that we allow for a stochastic environment that changes over time. This allows us to study the effects of good and bad states on signaling strategies. In the model of Saloner (1984), demand is uncertain, but demand shocks that last for a single period serve solely as a device to add noise to the incumbent's actions; the market conditions for both payers are identical before each round. Mester (1992) analyzes a three-period signaling setting, in which the unobservable type changes over time. Sweeting et al. (2020) also analyze a discrete-time, finite-horizon model with changing (cost) types and stationary demand. Aside from modeling choices, a major difference between their paper and ours is that they focus on fully separating equilibria, and we focus on equilibria with a combination of pooling and partial separation. Contrary to these papers, we assume that the *observable* market conditions fluctuate, but the unobservable type of the informed player is fixed. This also sets our paper apart from Toxvaerd (2017), who studies finitely repeated limit pricing with constant market conditions, but with various time horizons, in addition to Kaya (2009), who studies separating equilibria in an infinitely repeated (and, hence, with constant conditions) discrete-time signaling game.

Closer to our paper is Gryglewicz and Kolb (2022), which studies a dynamic signaling game, in which an informed player's payoffs depend on both her reputation and a public, stochastic stakes variable, and, in equilibrium, this player's strategy depends on the current stakes and their historical minimum. The key difference is that there is no strategic second player corresponding to our entrant in that paper. As a result, the strategic interaction between the players and the effect of timing decisions on outcomes is not explored in Gryglewicz and Kolb (2022). In contrast, the current paper presents a market equilibrium with endogenous prices and competition. This allows us to derive unique implications for price and entry dynamics. Finally, the modeling environment is different, as time is discrete in Gryglewicz and Kolb (2022).

Mixed strategies play an important role in our paper and in other dynamic stochastic games, but for different reasons. In stochastic contests and wars of attrition (Seel and Strack 2016, Georgiadis et al. 2022), players trade off the cost of continuing with the benefit of winning in case the other player quits sooner. Mixing by each player is calibrated to make the rival player indifferent. For instance, Georgiadis et al. (2022) show that, in their model, if flow payoffs are stochastic, the existence of mixed-strategy equilibria requires that players have identical exit payoffs. Mixing also induces indifference for rival players in stochastic preemption games (Riedel and Steg 2017). In contrast, in our paper, only the incumbent mixes, and the role of her mixing is to calibrate via Bayes' rule her reputation (conditional on continuing) to make *herself* indifferent. The need for this calibration stems from a negative-feedback effect: If the conjectured probability of revelation is too large, the weak incumbent can earn a high reputation by continuing, undermining her incentive to reveal in the first place.

Whereas costly signaling in a stochastic environment drives the reputation building in our model, other papers have modeled reputation building with the ability to make direct investments into quality; see Board and Meyer-ter-Vehn (2013), Dilmé (2019b), and Kolb (2019) with binary quality and Cisternas (2018) and Bohren (2023) for continuous quality.

Our work also relates to the literature on continuous-time games with an underlying diffusion process and, especially, to some papers that use continuous-time methods to solve dynamic information problems. Daley and Green (2012, 2020), McClellan (2022), and Kolb (2019) feature, like our paper, a privately informed player of binary type, but the underlying diffusion process is exogenous news about that type, giving direct rise to reputation dynamics; in contrast, the underlying diffusion process in our paper is the stakes or market size, and reputation dynamics emerge endogenously. In those papers, with the exception of McClellan (2022), the natural state variable has a single component, the informed player's current reputation. Bonatti et al. (2017) study dynamic Cournot competition among firms with private information about their marginal costs; their game is related to the postentry game in our model, which we simplify, in order to focus on the pre-entry game, by assuming that the incumbent's type gets immediately revealed at entry. Orlov et al. (2020) model a sender who wants to influence the time at which a receiver exercises a real option; there, the sender's strategic instrument is dynamic information design, whereas in our model, it is costly signaling.

The next section sets up the model as a signaling game and gives an informal preview of the equilibrium. Section 3 presents the equilibrium analysis of the model and discusses the main limit pricing application; the reader may read Section 3.5 immediately for microfoundations of the signaling model and then return to Section 2 without loss of continuity. Section 4 discusses various modeling assumptions and how a variation of the model can be used to study predatory pricing. Section 5 concludes. Proofs are contained in Appendices A–D and the electronic companion.

2. The Model

In this section, we set up the model as a signaling game between an entrant and an incumbent and give a preview of the equilibrium result. The modeling choices are microfounded in Section 3.5.

2.1. The Setup

The game takes place in continuous time over an infinite horizon, indexed by $t \in [0, \infty)$. Player 1 ("she") is of a privately known type (weak or strong) and wants to convince Player 2 ("he") that she is strong. Both players face optimal stopping problems in light of publicly evolving market demand: Player 1 (if weak) chooses when to reveal her type, and Player 2 chooses when to enter the market.

Uncertainty is modeled on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, defined as a product as follows. First, there is a standard Brownian motion W defined on a canonical probability space $(\Omega^W, \mathcal{F}^W, \mathbb{P}^W)$. Second, there is a probability space $(\Theta, 2^\Theta, \nu)$ for Player 1's type, where $\Theta = \{w, s\}$ and $\nu(\theta = S) = \pi_{0-} \in [0,1]$ is the entrant's prior. Third, there is a probability space $([0,1], \mathcal{L}, \lambda)$ to serve as Player 1's "randomization device." We then define $\Omega = \Omega^W \times \Theta \times [0,1]$, $\mathcal{F} = \mathcal{F}^W \times 2^\Theta \times \mathcal{L}$, and $\mathbb{P} = \mathbb{P}^W \times \nu \times \lambda$. A typical element of Ω is written (ω, θ, ζ).

The market size, or "stakes," evolves exogenously over time, according to a publicly observed geometric Brownian motion[2]

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

with $X_0 = x_0 > 0$. The constants $\mu \in \mathbb{R}$ and $\sigma > 0$ are drift and volatility parameters.

Player 2 and the weak type of Player 1 face interrelated stopping problems; the strong type of Player 1 is assumed to be nonstrategic and does not take any explicitly modeled actions. The weak type of Player 1 chooses a time ρ at which to publicly reveal her type, and Player 2 chooses a time τ to enter the market.[3] We assume that Player 2 observes Player 1's time-t decision before acting at time t, but not vice versa.

Table 1. Coefficients on Flow Payoffs in the Game Before and After Entry

Player	Before entry	After entry
Player 1, type w	S or M^w	D_1^w
Player 2	0	D_2^w or 0

Notes. The payoffs of Player 2 after entry are net of lump-sum cost K and depend on the type of Player 1 (D_2^w if $\theta = w$ or zero if $\theta = s$). It is assumed that $M^w > S > D_1^w$ and $D_2^w > 0$.

When Player 2 enters, he pays a lump-sum cost $K > 0$, and Player 1's type is immediately revealed. Thereafter, Player 2 earns a duopoly flow payoff of $D_2^w X_t$ if Player 1 is weak and zero if Player 1 is strong, whereas the weak type of Player 1 earns a duopoly flow payoff of $D_1^w X_t$. These postentry flow payoffs induce termination payoffs for the pre-entry stopping game. Prior to entry, Player 2 earns zero flow payoff, whereas the weak type of Player 1 earns a "signaling" flow payoff of SX_t before she reveals and monopoly flow payoff $M^w X_t$ after she reveals.

To make the game interesting while maintaining tractability, we make the following parametric assumptions. First, entry is never profitable against type s, but is profitable against type w if the stakes are sufficiently high to offset the fixed costs: $D_2^w > 0$. Second, continuing to signal is costly for type w: $M^w > S$. Third, type w would prefer signaling to facing immediate entry: $S > D_1^w$. Table 1 summarizes these payoff coefficients.

Both players are risk-neutral and discount payoffs at a constant rate r. We assume $r > \mu$ to guarantee finite discounted expected payoffs.

2.1.1. Strategies.

Let $\mathcal{F}^X = (\mathcal{F}_t^X)_{t \geq 0}$ denote the augmented filtration of the underlying probability space generated by X; $\mathcal{F}_t^X$ represents the public history of stakes up to time t.

A (mixed) strategy for Player 1 is a $\mathcal{F}^X$-measurable function ρ from Ω to $\mathbb{R}_+ \cup \{+\infty\}$ such that (i) $\rho(\cdot, s, \cdot) = +\infty$ (i.e., the strong type never stops), and (ii) $\rho(\cdot, w, \zeta)$ is a stopping time with respect to $\mathcal{F}^X$ for all $\zeta \in [0,1]$.[4] Note that mixed strategies are mappings from Player 1's randomization device into pure strategies (stopping times). Henceforth, we suppress the dependence of ρ on ω. We refer to $\rho(w, \cdot)$ as type w's strategy, and, in some abuse of notation, we will often use ρ to refer to a generic strategy for type w. A pure strategy for Player 1 type w is a stopping time with respect to $\mathcal{F}^X$.

To specify the information available to Player 2, define $\phi_t := \mathbb{1}\{\rho \leq t\}$, and define $\mathcal{F}^2 = (\mathcal{F}_t^2)_{t \geq 0}$ as the augmented filtration generated by (X, ϕ). A pure strategy for Player 2 is a stopping time τ with respect to $\mathcal{F}^2$.

Given a conjectured strategy ρ^* of Player 1, the history of the market, and any revelation by Player 1, Player 2 updates beliefs at time t using Bayes' rule whenever possible as follows. Define a nondecreasing, $[0,1]$-valued, right-continuous stochastic process R by $R_t = \mathbb{P}(\rho^*$

$\le t | \mathcal{F}_t^X, \theta = w)$, $t \ge 0$, representing the pathwise cumulative probability that Player 1 of the weak type reveals by time t. Then, given an *actual* strategy ρ for Player 1, Player 2's posterior belief that Player 1 is strong is $\pi_t = \pi_{0-}/[\pi_{0-} + (1-\pi_{0-})(1-R_t)]$ if $\pi_{0-} > 0$ and $\rho > t$, and $\pi_t = 0$ if $\rho \le t$ or $\pi_{0-} = 0$. It is always on path from the perspective of Player 2 for Player 1 not to have revealed yet if $\pi_{0-} > 0$. If $\pi_{0-} = 0$ and $R_t = 1$, but $\rho > t$ (i.e., the belief was at zero and Player 1 should have revealed already, but has not), then Bayes' rule does not apply, and we impose that the belief remains at zero forever. In addition, once Player 1 reveals, the belief jumps to zero, even if it starts at $\pi_{0-} = 1$, so that revelation is off-path.

It is useful to work with log-likelihood beliefs $Z_t = \ln(\pi_t/(1-\pi_t))$, $t \ge 0-$, which follow

$$Z_t = \begin{cases} Z_{0-} - \ln(1-R_t) & \text{if } Z_{0-} > -\infty \text{ and } \rho > t \\ -\infty & \text{if } \rho \le t \text{ or } Z_{0-} = -\infty. \end{cases} \quad (1)$$

We focus on equilibria, in which behavior depends on the history only through the pair (X, Z). A Markov strategy for Player 1 is ρ such that $(X_t, Z_t)_{t \ge 0-}$ is a time-homogeneous $\mathcal{F}^2$-Markov process, where $X_{0-} := X_0$. We say that τ is a Markov strategy for Player 2 if there exists $\mathcal{D} \subset \mathbb{R}_{>0} \times \bar{\mathbb{R}}$, where $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$, such that $\tau = \inf\{t > 0 : (X_t, Z_t) \in \mathcal{D}\}$.

2.1.2. Payoffs. Let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$, and let $\mathbb{P}_{x,z}$ and $\mathbb{E}_{x,z}$ denote conditional probabilities and expectations starting from $(X_0, Z_{0-}) = (x, z)$. Given a conjectured strategy τ of Player 2, Player 1 type w's expected payoff from a strategy ρ starting from (x, z) is

$$U_1(\rho, \tau; x, z) = \mathbb{E}_{x,z}\left[\int_0^{\rho\wedge\tau} e^{-rt} S X_t\, dt + \mathbb{1}\{\rho \le \tau\} \int_\rho^\tau e^{-rt} M^w X_t\, dt + e^{-r\tau} \frac{D_1^w}{r-\mu} X_\tau\right]. \quad (2)$$

Given a conjectured strategy ρ of Player 1, Player 2's expected payoff is

$$U_2(\tau, \rho; x, z) = \mathbb{E}_{x,z}\left[e^{-r\tau}\left(\mathbb{1}\{\theta = w\} \frac{D_2^w}{r-\mu} X_\tau - K\right)\right]. \quad (3)$$

Definition 1. A Markov equilibrium is a pair of Markov strategies (ρ, τ) and a belief process Z such that

1. (Player 1 Optimality). For all $\zeta \in [0,1]$ and all strategies ρ' for Player 1, $U_1(\rho(\zeta), \tau; x, z) \ge U_1(\rho', \tau; x, z)$ for all $(x, z) \in \mathbb{R}_{>0} \times \bar{\mathbb{R}}$.
2. (Player 2 Optimality). For all pure strategies τ' for Player 2, $U_2(\tau, \rho; x, z) \ge U_2(\tau', \rho; x, z)$ for all $(x, z) \in \mathbb{R}_{>0} \times \bar{\mathbb{R}}$.
3. (Bayesian Consistency). Z satisfies (1).

Our notion of Markov equilibrium is very similar to the notion of Markov perfect equilibrium in Orlov et al. (2020). Markov (perfect) equilibria are typically defined as either perfect Bayesian equilibria or Nash equilibria, with the additional property that strategies depend on histories only through a set of natural state variables. We use a static (Nash) equilibrium concept and refrain from using the "perfect" qualifier—defining strategies and optimality beginning at time 0—only to avoid the notational burden of explicitly specifying continuation behavior after every history, including players' own deviations. However, the continuation behavior implied by our Markov strategies is sequentially rational.

2.2. Equilibrium Description

To motivate our equilibrium construction, suppose Player 1 continues without revealing over an interval during which the weak type would reveal with positive probability. Player 2 revises his belief upward in favor of the strong type. Consequently, Player 2 must wait for the stakes to reach higher levels for entry to become profitable. In turn, Player 1 of the weak type now has a more valuable reputation at stake and is, therefore, willing to continue without revealing at lower stakes than before.

Building on this intuition, we construct a Markov equilibrium in cutoff strategies that can be characterized by two functions, U strictly increasing and L strictly decreasing, of Player 2's posterior belief that Player 1 is strong:

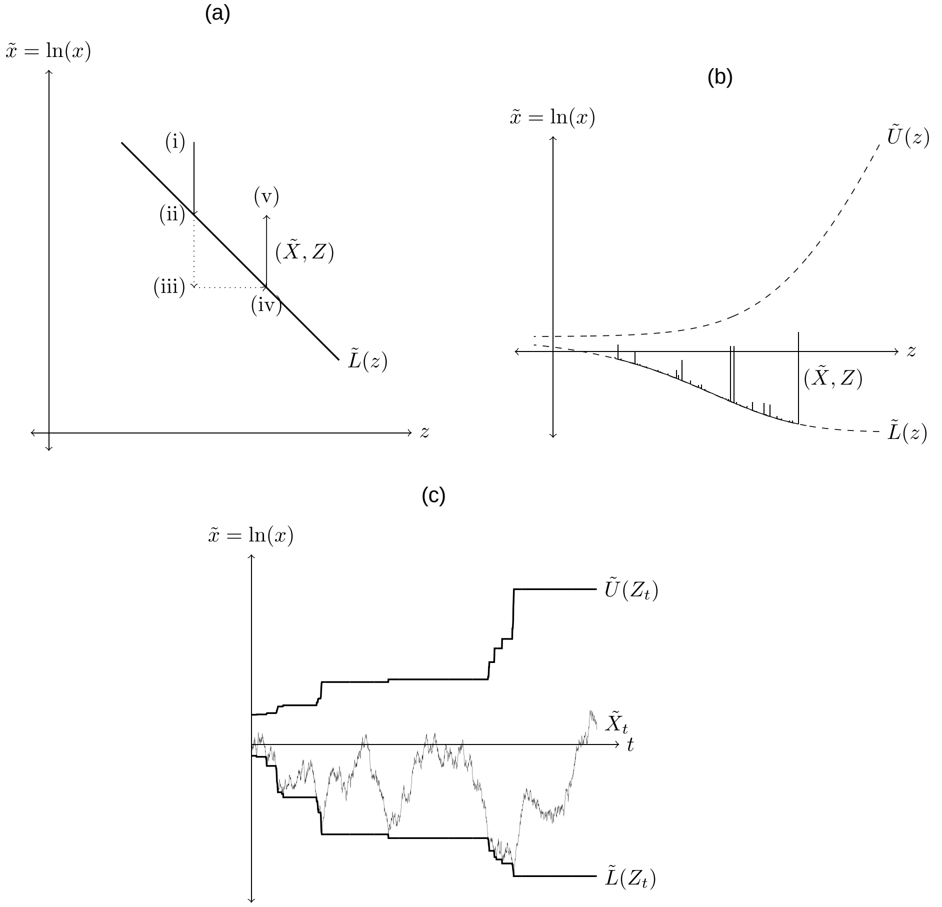
- Player 2 enters at time $\tau^U := \inf\{t \ge 0 : X_t \ge U(Z_t)\}$.
- Player 1, type w reveals at time $\rho^L := \inf\{t \ge 0 : X_t \le x^L(\zeta)\}$, where $x^L(\zeta) := L(Z_{0-} - \ln(1-\zeta))$ and where $L(+\infty) := \inf_{z\in\mathbb{R}}(L(z))$.

In words, Player 2 enters the first time X exceeds $U(Z)$. Meanwhile, Player 1 plays a mixed strategy. She continues whenever $X_t > L(Z_t)$ and mixes over revealing whenever $X_t = L(Z_t)$, so that the curve $\{(L(z), z) : z \in \mathbb{R}\}$ serves as a reflecting boundary for the process (X_t, Z_t), conditional on Player 1 continuing to signal. If $X_t < L(Z_t)$, type w either (i) mixes by revealing with an atom of probability such that, conditional on continuing, (X_t, Z_t) immediately returns to the curve $\{(L(z), z) :\ z \in \mathbb{R}\}$; or (ii) reveals with certainty if $X_t \le L(z)$ for all $z \in \mathbb{R}$. Because L is decreasing, it follows that, once the stakes first reach $L(z)$, Player 1's reputation increases whenever the market reaches a new minimum, conditional on her continuing.

Definition 2. A (U, L) equilibrium is a pair of C^2 functions $U, L : \mathbb{R} \to (0, \infty)$, where $U > L$, U is increasing, and L is decreasing, such that (τ^U, ρ^L) and Z defined by (1) is a Markov equilibrium.

Figure 1 shows a sample path of the state variable process in equilibrium, putting log-market size $\tilde{X} = \ln(X)$ on the vertical axis.[5] Note that during intervals of time for which $\tilde{X}_t > \tilde{L}(Z_t)$, both Player 1 types are pooling, and, thus, the belief process Z_t is constant at some z, and the process $(\tilde{X}_t, Z_t) = (\tilde{X}_t, z)$ moves only in the vertical dimension. If $\tilde{X}_t$ reaches $\tilde{U}(z)$ (not shown), Player 2 immediately enters the market, and the game effectively ends. If $\tilde{X}_t$ reaches $\tilde{L}(z)$, type w reveals

Figure 1. A Sample Path of $(\tilde{X}_t, Z_t)$ in a (U, L) Equilibrium



Notes. (a) A heuristic sample path given by a discrete random walk. Starting from (i), there is a negative shock to $\tilde{X}$ leading to (ii); another negative shock leads to (iii), where, conditional on not revealing, belief updating leads to (iv). A positive shock to $\tilde{X}$ then leads to (v). (b) An actual continuous-time sample path in (x, z) space. (c) The evolution of $(\tilde{X}_t, \tilde{U}(Z_t), \tilde{L}(Z_t))$ over time.

gradually, so that if w does *not* reveal, Player 2 revises his belief slightly upward, causing the $(\tilde{X}_t, Z_t)$ process to remain on or above the $\tilde{L}(z)$ curve.[6]

Our main result (Theorem 1) establishes that there exists a unique (U, L) equilibrium, and we characterize the equilibrium signaling and entry strategies as solutions to ordinary differential equations. We highlight the main intuition here; a detailed analysis is in Section 3. There, we also show that in any Markov equilibrium, players' strategies involve cutoffs with respect to stakes as a function of beliefs.

Suppose that the current belief is some $z \in \mathbb{R}$. If the market is sufficiently large, then Player 2 should enter; the fixed costs of entry are small, relative to the upside to entering, in the event that Player 1 turns out to be weak. This entry threshold is an increasing function of z because, by definition, Player 1 is less likely to be weak when z is large. Next, we argue that the weak type must mix over revealing at some lower cutoff $L(z)$. The argument involves two steps: (i) ruling out the possibility that w never reveals, and (ii) ruling out the possibility that w reveals with probability one at some state. As for (i), suppose that type w instead *never* reveals. Then, starting from $(\tilde{X}_t, Z_t) = (\tilde{x}, z)$, because no information arrives about Player 1, the belief process Z_t holds constant at z. Now, for small x, there is a large expected delay until Player 2 is willing to enter, even if it were known that Player 1 were type w, because of the fixed cost of entry K. By discounting, the advantage to w of being perceived as type s approaches zero as x approaches zero. Type w would then strictly prefer to reveal in order to save the cost of signaling between now and when entry occurs, contradicting equilibrium. Thus, some revelation by the weak type must occur eventually; let $L(z)$ denote the highest stakes at which the weak type is willing to reveal.

Turning to (ii), we argue that the weak type cannot reveal with probability one at $L(z)$—she must mix at $L(z)$. If the weak type were conjectured to reveal with probability one at $L(z)$, then, by deviating and continuing to signal an extra instant, w could convince the entrant that she is strong and, thus, would strictly benefit, contradicting equilibrium. Thus, the weak type cannot reveal with probability one at $L(z)$, and because we have already argued that *some* revelation occurs at $L(z)$, w must mix at $L(z)$.

In equilibrium, the threshold functions U and L are codependent and must be determined together. First, to see how L depends on U, note that mixing requires w to be indifferent between revealing and continuing at $L(z)$, and, thus, the reputational benefit of continuing to signal must exactly offset the signaling cost. This reputational benefit depends on both the shape of U and the increase in Player 1's reputation when she continues to signal. Fixing U, the intensity of revelation in equilibrium is calibrated so that Player 1's increase in reputation by continuing, via Bayes' rule, induces indifference. This intensity of revelation is then manifest in the shape of L: A flatter L implies greater sensitivity of reputation to downward shocks in stakes at $L(z)$ and, thus, a larger probability of revelation. Second, and conversely, U depends on L: Player 2's entry threshold is influenced by the option value of delay, which depends on the amount of information Player 2 receives at $L(z)$ due to mixing by Player 1, type w.

3. Equilibrium Analysis

In this section, we construct a (U, L) equilibrium as introduced above, and we show that it is unique within the class of (U, L) equilibria.

3.1. Player 1 Believed to be Weak

In a (U, L) equilibrium, the value functions for both players depend on the current state (x, z). Let $F, V : \mathbb{R}_{>0} \times \bar{\mathbb{R}}$ denote the value function for Player 1 (type w) and Player 2, respectively.

Once a weak type reveals, we have $z = -\infty$, and the continuation game is a standard single-player stopping problem by Player 2. Because the continuation payoffs in this case serve as terminal payoffs for the game prior to revelation by w, it is useful to characterize them here.

Let $F_T(x) := F(x, -\infty)$ and $V_T(x) := V(x, -\infty)$ denote the value functions for Players 1 and 2, respectively, after Player 1 has revealed, where T is mnemonic for "termination." Player 2's strategy then is to enter at an upper threshold u_T when the payoff is large enough to compensate the fixed cost of entry.

For $x < u_T$, Player 2's value function satisfies

$$rV_T(x) = \mu x V_T'(x) + \frac{1}{2}\sigma^2 x^2 V_T''(x). \tag{4}$$

The left-hand side of (4) represents the required rate of return per unit of time, and the right-hand side is the expected change in continuation value. Here, and in what follows, we make repeated use of the fact that equations of the form

$$rh(x) = \mu x h'(x) + \frac{1}{2}\sigma^2 x^2 h''(x) + \psi x, \tag{5}$$

have general solution of the form $c_1 x^{\beta_1} + c_2 x^{\beta_2} + \psi x/(r - \mu)$, for unknown constants c_1 and c_2 to be pinned down by boundary conditions and known constants $\beta_1 > 1$ and $\beta_2 < 0$.[7] Thus, the general solution to (4) has the form

$$V_T(x) = B_1(-\infty)x^{\beta_1} + B_2(-\infty)x^{\beta_2}, \tag{6}$$

where our notation for the coefficients makes explicit their dependence on z via the boundary conditions.

Player 2's value function satisfies three boundary conditions: $V_T(u_T) = D_2^w u_T/(r - \mu) - K$, $V_T'(u_T) = D_2^w/(r - \mu)$, and $V_T(0+) = 0$. The first two of these are standard value-matching and smooth-pasting conditions associated with optimal stopping at u_T, and the third says that Player 2's option to enter becomes worthless as the market vanishes. These conditions pin down the stopping threshold

$$u_T = \frac{\beta_1 K(r - \mu)}{(\beta_1 - 1)D_2^w}, \tag{7}$$

and the value function

$$V_T(x) = \begin{cases} \left(\frac{x}{u_T}\right)^{\beta_1}\left(\dfrac{D_2^w u_T}{r - \mu} - K\right) = B_T x^{\beta_1}, & x < u_T \\ \dfrac{D_2^w x}{r - \mu} - K, & x \geq u_T, \end{cases} \tag{8}$$

where $B_T := D_2^{w\beta_1}(\beta_1 - 1)^{\beta_1 - 1}/(\beta_1^{\beta_1} K^{\beta_1 - 1}(r - \mu)^{\beta_1})$.

Knowing the optimal behavior of Player 2, we can characterize $F_T(x)$. For $x \geq u_T$, type w collects a flow payoff of $D_1^w X_t$ forever, with discounted value $D_1^w x/(r - \mu)$. For $x < u_T$, she collects flow payoff $M^w X_t$ until entry, and her value function solves the ordinary differential equation (ODE)

$$rF_T(x) = \mu x F_T'(x) + \frac{1}{2}\sigma^2 x^2 F_T''(x) + M^w x. \tag{9}$$

The general form of the solution to (9) is

$$F_T(x) = A_1(-\infty)x^{\beta_1} + A_2(-\infty)x^{\beta_2} + \frac{M^w x}{r - \mu}. \tag{10}$$

The constants $A_1(-\infty)$ and $A_2(-\infty)$ are pinned down by the boundary conditions $F_T(u_T) = D_1^w u_T/(r - \mu)$ and $F_T(0+) = 0$. These yield $A_1(-\infty) = A_T := (D_1^w - M^w)(u_T)^{1-\beta_1}/(r - \mu) < 0$ and $A_2(-\infty) = 0$, so that

$$F_T(x) = \begin{cases} A_T x^{\beta_1} + \dfrac{M^w}{r - \mu}x, & x < u_T \\ \dfrac{D_1^w}{r - \mu}x, & x \geq u_T. \end{cases} \tag{11}$$

3.2. Player 1 Believed to be Strong

We now analyze the case where Player 1 is perceived to be strong with certainty: $z = +\infty$. In this case, Player 2's continuation value is identically zero: $V(x, +\infty) = 0$, all $x \in \mathbb{R}$, and unless Player 1 reveals as type w, Player 2 never enters (i.e., $U(+\infty) = +\infty$).

However, w still reveals at some lower threshold $\underline{L}$ to be determined. When continuing, Player 1's value function satisfies

$$rF(x,z) = \mu x F_x(x,z) + \frac{1}{2}\sigma^2 x^2 F_{xx}(x,z) + Sx. \tag{12}$$

The general form of the solution to (12) is

$$F(x,z) = A_1(z)x^{\beta_1} + A_2(z)x^{\beta_2} + \frac{Sx}{r-\mu}. \tag{13}$$

For the case $z = +\infty$, the constants are pinned down by the following boundary conditions. First, as the market becomes arbitrarily large and far away from the stopping boundary $\underline{L}$, type w's continuation payoff converges to the value of continuing in monopoly forever:

$$\lim_{x\to+\infty}\left|F(x,+\infty) - \frac{Sx}{r-\mu}\right| = 0,$$

which implies $A_1(+\infty) = 0$. Second, given any $\underline{L}$, the boundary condition at $x = \underline{L}$ is $F(\underline{L}, +\infty) = F_T(\underline{L})$, and optimality implies the smooth pasting condition $F_x(\underline{L}+, +\infty) = F_T'(\underline{L})$. Putting these together, we obtain the cutoff

$$\underline{L} = \left[\frac{(1-\beta_2)(M^w - S)}{(\beta_1-\beta_2)(M^w - D_1^w)}\right]^{\frac{1}{\beta_1-1}} u_T < u_T,$$

and the continuation value

$$F(x,+\infty) = \begin{cases} A_2(+\infty)x^{\beta_2} + \dfrac{Sx}{r-\mu}, & x \geq \underline{L} \\ F_T(x), & x < \underline{L} \end{cases},$$

where $A_2(+\infty) = \underline{L}^{1-\beta_2}\frac{\beta_1-1}{\beta_1-\beta_2}\frac{M^w-S}{r-\mu}$.

3.3. Interior Beliefs

Now, we consider beliefs $z \in (-\infty, +\infty)$. When x is strictly above $L(z)$, type w continues with certainty, and, thus, the entrant learns nothing about the incumbent's type, conditional on no revelation occurring. The only dynamics in this case are with respect to market size, and we can characterize value functions along vertical cross-sections of the (x, z) space (where x is the vertical dimension).

Given z, the weak incumbent's value function for $x \in (L(z), U(z))$ satisfies (12) and has the general form $A_1(z)x^{\beta_1} + A_2(z)x^{\beta_2} + Sx/(r-\mu)$. Type w earns a termination payoff $D_1^w U(z)/(r-\mu)$ when Player 2 enters at $U(z)$. By revealing at $L(z)$, she earns a termination payoff of $F_T(L(z))$, characterized in the previous section. These termination payoffs yield the following two value-matching conditions:

$$F(U(z),z) = \frac{D_1^w U(z)}{r-\mu}, \tag{14}$$

$$F(L(z),z) = F_T(L(z)). \tag{15}$$

Because of Brownian noise and the offsetting upward revision in beliefs given by (1) in response to downward shocks along the $L(z)$ curve, we also have the smooth fit and normal reflection conditions[8]

$$F_x(L(z),z) = F_T'(L(z)), \tag{16}$$

$$F_z(L(z),z) = 0. \tag{17}$$

Note that total differentiation of (15) (which holds for all $z \in \mathbb{R}$) with respect to z shows that (16) implies (17).

Using (11) and (13), we solve (15) and (16) for $A_1(z)$ and $A_2(z)$ in terms of $L(z)$:

$$A_1(z) = A_T + \frac{1-\beta_2}{\overline{\beta}}\frac{M^w - S}{r-\mu}L(z)^{1-\beta_1},$$

$$A_2(z) = \frac{\beta_1 - 1}{\overline{\beta}}\frac{M^w - S}{r-\mu}L(z)^{1-\beta_2}, \tag{18}$$

where $\overline{\beta} := \beta_1 - \beta_2$.

Substituting these into (14), dividing through by $U(z)$, recalling the log transformations $\tilde{U}(z) := \ln(U(z))$ and $\tilde{L}(z) := \ln(L(z))$, and defining $\Delta = \tilde{U} - \tilde{L}$, we get the following equation:

$$\begin{aligned} G(\tilde{U},\Delta) := {} & \frac{1-\beta_2}{\overline{\beta}}\frac{M^w - S}{r-\mu}e^{(\beta_1-1)\Delta} \\ & + \frac{\beta_1-1}{\overline{\beta}}\frac{M^w-S}{r-\mu}e^{(\beta_2-1)\Delta} \\ & + A_T e^{(\beta_1-1)\tilde{U}} + \frac{S - D_1^w}{r-\mu} = 0. \end{aligned} \tag{19}$$

As stated in Lemma 1, Equation (19) allows us to define a "best response" threshold curve $\tilde{L}(z)$ for the incumbent to a given threshold curve $\tilde{U}(z)$ of the entrant. In particular, we argue that U and L satisfy a one-to-one relationship for each z; this will allow us to reduce the system to a single ODE with respect to U. As U increases, all else equal, the weak type of Player 1 has more to lose by revealing and, thus, waits longer to reveal (L decreases in U). In Lemma 1, we use $f(\tilde{U})$ to denote the value of Δ, given $\tilde{U}$, that solves (19); given z, $f(\tilde{U}(z))$ is the length of the pooling interval, and, intuitively, it measures the signaling effort that the weak incumbent is willing to exert in order to keep the entry threshold at $\tilde{U}(z)$, rather than at $\tilde{u}_T$.

Lemma 1. *Equation* (19) *implicitly defines a function* $f: [\tilde{u}_T, +\infty) \to \mathbb{R}_+$ *so that* $G(\tilde{U}, f(\tilde{U})) = 0$. *The function* f *is increasing and continuously differentiable,* $f(\tilde{u}_T) = 0$, $f'(\tilde{U}) > 1$ *for all* $\tilde{U} > \tilde{u}_T$, *and* $\lim_{\tilde{U}\to\infty} f'(\tilde{U}) = 1$. *Thus,* (19) *defines a strictly decreasing, differentiable best response* $\tilde{L}(\tilde{U})$.

The analysis above characterizes $F(x, z)$ for $x \in [L(z), U(z)]$ given U. To complete the characterization, we specify Player 1's value function outside this interval. For $x > U(z)$, Player 1 anticipates a flow payoff of $D_1^w X_t$ forever, for a discounted value of $D_1^w X_t/(r-\mu)$. For $x \leq L(z)$, Player 1 weakly prefers to reveal for a termination payoff of $F_T(x)$.

We now characterize Player 2's equilibrium strategy and value function. For $x \in [L(z), U(z)]$, the value function satisfies the same ODE as in (4), and the solution has the form $V(x,z) = B_1(z)x^{\beta_1} + B_2(z)x^{\beta_2}$. The boundary conditions are

$$V(U(z),z) = \frac{D_2^w U(z)}{(1+e^z)(r-\mu)} - K, \tag{20}$$

$$V_x(U(z),z) = \frac{D_2^w}{(1+e^z)(r-\mu)}, \tag{21}$$

$$V_z(L(z),z) = \frac{1}{1+e^z}[V(L(z),z) - V_T(L(z))], \tag{22}$$

where $1/(1+e^z) = 1-\pi$. Again, Equations (20) and (21) are value-matching and smooth-pasting conditions, respectively, associated with optimal stopping for Player 2. Equation (22) is a Robin condition, which says that the marginal decrease in Player 2's continuation value, which occurs when Player 1 continues signaling, must exactly offset, in expectation, a discrete gain in continuation value that occurs when Player 1 is weak and reveals herself. Using (6), (20), and (21), we get closed-form expressions for $B_1(z), B_2(z)$ in terms of $\tilde{U}(z)$, which are given as (B.1) and (B.2) in Appendix B.

Substituting these into (22) yields an ordinary differential equation for $\tilde{U}$ (see (B.3)). Together, (19) and (B.3) form a system of equations for $(\tilde{U}, \tilde{L})$ and, thus, (U, L).

Given $\tilde{L}$, the maximality principle of Peskir (1998) pins down $\tilde{U}$. However, this is not sufficient for obtaining a unique equilibrium candidate, because $\tilde{L}$ itself depends on $\tilde{U}$. Therefore, we reduce this system for $(\tilde{U}, \tilde{L})$ to a single equation for $\tilde{U}$ and, in the next section, invoke an antifunnel argument to obtain a unique solution. Following Lemma 1, Equation (B.3) can be written as

$$\tilde{U}' = \frac{\begin{array}{l}[e^{\overline{\beta} f(\tilde{U})}(D_2^w(\beta_1-1)e^{\tilde{U}} - \beta_1 K(r-\mu)) - \overline{\beta} B_T(r-\mu)e^{\beta_1 \tilde{U}} \\ + D_2^w(1-\beta_2)e^{\tilde{U}} + \beta_2 K(r-\mu)]\end{array}}{(e^{\overline{\beta} f(\tilde{U})}-1)(D_2^w(\beta_1-1)(1-\beta_2)e^{\tilde{U}} + \beta_1\beta_2 K(r-\mu)(1+e^z))} =: g(\tilde{U}, z). \tag{23}$$

Given a solution $\tilde{U}$ to (23), $\tilde{L}$ is determined by (19).

To complete the characterization of Player 2's candidate value function, note that for $x \geq U(z)$, he enters immediately and obtains a value of $D_2^w x/[(1+e^z)(r-\mu)] - K$. For $x < L(z)$, it is unnecessary to specify the value, because the state immediately exits this region in one of the three ways: (i) Player 1 reveals immediately as weak; (ii) Player 1 does not reveal, and the belief immediately jumps to z' so that $x = L(z')$; or (iii) Player 1 does not reveal, and the belief jumps to $+\infty$.

3.4. Existence and Uniqueness

In this section, we establish the existence and uniqueness of a (U, L) equilibrium for the game and discuss the qualitative features of the signaling and entry strategies. In particular, we sketch the main arguments for the existence and uniqueness of a solution to (23) consistent with equilibrium. We exploit the economic forces at play: The entrant's option value depends on the extent of the incumbent's information revelation at low stakes.

Because revelation by weak types at a lower threshold gives the entrant option value from delaying entry, a lower bound on the entrant's threshold is the best response when he believes Player 1 of the weak type never reveals (which implies that Z is constant). This lower bound can be expressed in closed form:

$$\tilde{U}_0(z) := \ln\left(\frac{\beta_1 K(r-\mu)}{(\beta_1 - 1)D_2^w}(1+e^z)\right). \tag{24}$$

Note that as $z \to -\infty$, $\tilde{U}_0(z) \to \tilde{u}_T$.

The next lemma says that the right side of (23) satisfies a Lipschitz condition above $\tilde{U}_0$, guaranteeing local existence and uniqueness of a solution to (23) for any initial condition above $\tilde{U}_0$.

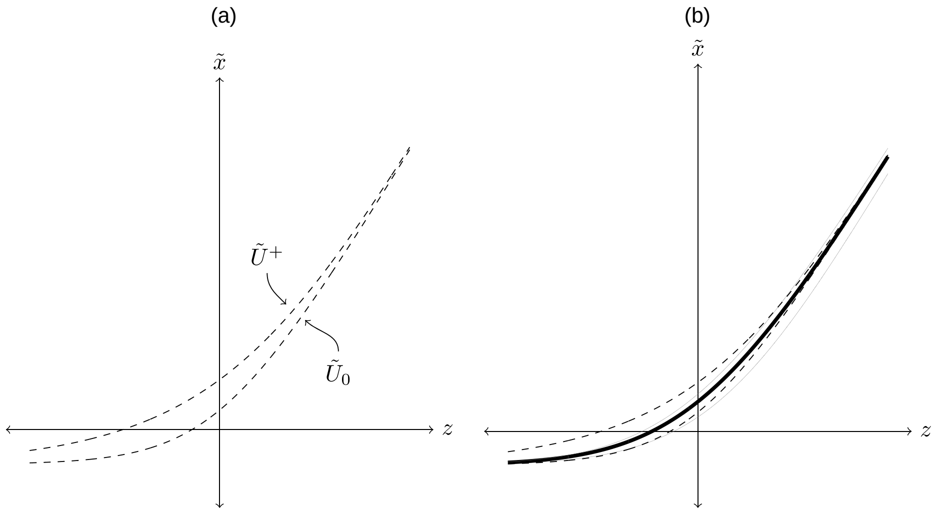
Lemma 2. *The function g on the right side of (23) is Lipschitz continuous in $\tilde{U}$ uniformly in z in the domain $\{(\tilde{U}, z) : z \in \mathbb{R}, \tilde{U} \geq \tilde{U}_0(z)\}$, and, in this domain, $\tilde{U}'(z) > 0$ for any solution to (23).*

Nonetheless, there will exist many global solutions to (23) that do not correspond to equilibria. In addition to the lower bound $\tilde{U}_0$, it is useful to construct an upper bound, denoted $\tilde{U}^+(z)$, on the entrant's threshold that is the best response to an incumbent who fully separates at $\tilde{u}_T$, maximizing the information the entrant can obtain by waiting. We provide the derivation of $\tilde{U}^+(z)$ in the Appendix B. Figure 2(a) illustrates these upper and lower bounds.

Lemma 3. *In any (U, L) equilibrium, $\tilde{U}_0(z) \leq \tilde{U}(z) \leq \tilde{U}^+(z)$ for all $z \in \mathbb{R}$.*

Hence, we wish to find a unique global solution to (23) satisfying the inequalities in Lemma 3. To that end, we use the concept of an *antifunnel* from the theory of differential equations (Hubbard and West 1991, p. 31). We show in Lemma 4 that $\tilde{U}^+$ and $\tilde{U}_0$ form an antifunnel for (23) in that (i) any solution that crosses $\tilde{U}^+$ must do so from below and then stay above it; and (ii) any solution that crosses $\tilde{U}_0$ must do so from above and then stay below $\tilde{U}_0$. We then show that these antifunnel bounds converge sufficiently fast, such that there is a unique global solution $\tilde{U}$, which lies entirely inside the antifunnel. Figure 2(b) illustrates the antifunnel. The dark solid curve is the unique solution $\tilde{U}$ inside it, whereas the light solid curves are other solutions, all of which escape the antifunnel either through $\tilde{U}^+$ or through $\tilde{U}_0$.[9]

Figure 2. The Functions $\tilde{U}^+$ and $\tilde{U}_0$ Form an Antifunnel for the Differential Equation (23)



Notes. (a) Upper and lower bounds. (b) Solutions to ODE.

Lemma 4. *The functions $\tilde{U}^+$ and $\tilde{U}_0$ form an antifunnel for (23) and converge sufficiently fast that there is a unique solution $\tilde{U}$ that stays in the antifunnel, and it satisfies $\tilde{U}(z) \in (\tilde{U}_0(z), \tilde{U}^+(z))$ for all $z \in \mathbb{R}$.*

Putting together these results with a final verification of optimality, we have the following.

Theorem 1. *There exists a unique (U, L) equilibrium, and the signaling and entry thresholds are solutions to (19) and (23).*

In summary, we have shown that Player 1's strategy involves randomizing over revealing at the lower trigger $L(z)$, which is decreasing in z. Separation occurs gradually, except for an atom at time 0 in case the starting state is below $L(z)$. Player 2's strategy is to enter at the upper trigger $U(z)$, increasing in z. The threshold $L(z)$ is such that the weak type is indifferent between revealing versus paying the signaling cost over the next instant in order to marginally increase her reputation and, thereby, deter entry further, should the market rise. Player 2's strategy trades off the benefit of immediate entry with the option value of entering later in a stronger market or, in the event that the market falls to L, for the incumbent to reveal more information by partial separation.

Theorem 1 establishes uniqueness within the class of (U, L) equilibria, which is a subclass of Markov equilibria. This raises questions of whether (U, L) strategies are robust features of the signaling game that we consider. The following result, proved in the electronic companion, establishes that any Markov equilibrium has a cutoff structure characterized by two thresholds U and L (not necessarily C^2 or monotone), where $U > L$.

Proposition 1. *In any Markov equilibrium, there exist real-valued functions U and $L < U$ such that (i) Player 2 enters immediately when $x \geq U(z)$, and (ii) revealing is (weakly) optimal for type w of Player 1 when $x \leq L(z)$.*

3.5. Microfoundation

In this section, we add more structure to the model and provide a microfoundation for the reduced-form signaling game presented in Section 2.

There is a flow of demand so that the market clears at each time t. We assume that the demand function is isoelastic and subject to stochastic shocks. Specifically, the inverse demand function of total flow output Q at time t is given by

$$P_t(Q) = Y_t Q^{-\frac{1}{\gamma}},$$

where $\gamma > 1$ is the elasticity of demand and where the process $Y = (Y_t)_{t \geq 0}$ is a geometric Brownian motion with drift μ_Y and volatility σ_Y driven by a standard Brownian motion W:

$$dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dW_t.$$

Denoting the incumbent's (private) cost type by $\theta \in \{w, s\}$, the incumbent's production technology either has low (constant) marginal cost C_1^s or high cost C_1^w per unit of time, with $0 < C_1^s < C_1^w$. The entrant's marginal cost $C_2 > 0$ is common knowledge. We assume that after entry, the incumbent's type is revealed, and the two firms compete in quantities in Cournot fashion, repeating the static Nash equilibrium at each instant.[10]

It is straightforward to derive that the optimal production and profit flow of an unconstrained (that is, not facing a potential entrant) monopolist of type θ in each state Y_t are

$$Q_t^\theta = Y_t^\gamma \left(\frac{\gamma C_1^\theta}{\gamma - 1}\right)^{-\gamma}, \qquad \Pi_t^\theta := \frac{Y_t^\gamma}{\gamma}\left(\frac{\gamma C_1^\theta}{\gamma - 1}\right)^{1-\gamma}. \tag{25}$$

We now show that the profit flow Π_t^θ is itself a geometric Brownian motion, consistent with the original model.

Define a new variable $X_t = f(Y_t) = (Y_t)^\gamma$. By Itô's lemma, we have

$$dX_t = f'(Y_t)dY_t + \frac{1}{2}f''(Y_t)dY_t^2 = \mu X_t dt + \sigma X_t dW_t,$$

where $\mu := \gamma\mu_Y + \frac{1}{2}\gamma(\gamma-1)\sigma_Y^2$ and $\sigma := \gamma\sigma_Y$ are constants. Therefore, X is also a geometric Brownian motion adapted to the same filtration. Note that $\gamma^{-1}(\gamma C_1^\theta/(\gamma-1))^{1-\gamma}$ in (25) is constant over time; thus, the profit flow in (25) may be written as $M^\theta X_t$, as in Section 2, where $M^\theta = \gamma^{-1}(\gamma C_1^\theta/(\gamma-1))^{1-\gamma}$.

Similarly, the profit flow of a type θ incumbent in monopoly imitating the monopoly strategy of a firm with cost $\tilde{C}$ has the form $M^\theta(\tilde{C})X_t$; its profit flow in duopoly $D_1^\theta X_t$; and firm 2's profit flow after entry against type θ $D_2^\theta X_t$.[11]

We make the following assumptions about the cost structure: (i) There is profitable entry into the market against the w type: $D_2^w > 0$; (ii) the incumbent of type w prefers imitating the s-type to facing immediate entry: $M^w(C_1^s) > D_1^w$; and (iii) the incumbent of type s is a natural monopolist in that $D_1^s = M^s$ (whereas $D_2^s = 0$).[12] Assumption (i) rules out uninteresting equilibria, in which the weak type of Player 1 reveals itself at time 0, and Player 2 never enters. Assumption (ii) is a necessary condition for the incumbent of type w to engage in signaling.

Assumption (iii) reduces multiplicity by pinning down the strong type's behavior in the pre-entry game and allows us to model the dynamic signaling game with a binary action. As in simple one-shot signaling games, in principle, a continuum of prices may be sustained in equilibrium. We circumvent this multiplicity with assumption (iii), ensuring that the Cournot duopoly outcome with a realized strong incumbent entails zero production by the entrant and monopoly profits for the incumbent. As a consequence, prior to entry, the strong incumbent type has neither a signaling motive (because she has no need to deter entry) nor a short-run profit motive to deviate from her monopoly price. An implication is that any deviation from this price amounts to revelation for type w, and, thus, we restrict her to binary choices.

3.6. Implications

By introducing continuous-time dynamics and uncertainty, we can derive some interesting implications that are unavailable in the existing game-theoretic models of limit pricing. We formulate here several observations that are direct consequences of the equilibrium result and are of interest as empirical predictions.

Under our assumption of isoelastic demand and constant marginal cost, (unconstrained) monopoly prices are constant. But, in our equilibrium, entry deterrence results in price dynamics, and, moreover, the price can *increase* over an interval of time during which demand falls, as formally stated in Proposition 2. This is due to the incumbent revealing its type at a random lower trigger by adjusting its price discretely upward. This observation provides an empirical prediction that could be taken to the data, a prediction that would be difficult to obtain from a one-shot model.

Proposition 2 (Price Dynamics). *Fix $Z_{0-} \in \mathbb{R}$ and $X_0 \in (L(Z_{0-}), U(Z_{0-}))$. With positive probability, there exists a time $t > 0$ such that $P_t > P_0$, while $Y_t < Y_0$.*

Our model shows that market dynamics (that is, in our setup, the transition from monopoly to duopoly) exhibit path dependence in that the entrant's decision to enter depends on historical demand, namely, through its historical minimum; Figure 1(c) illustrates. This is despite the fact that the demand shocks are Markovian and the current demand level is a sufficient statistic for its distribution at future dates. Because a market downturn in the past makes it more likely that the weaker type of incumbent would have stopped using limit pricing, the probability of facing the strong incumbent increases under the limit-pricing regime. In other words, a demand slump polarizes entry timing—entry occurs either early against the revealed weak incumbent or late against an uncertain type. More broadly, our results suggest that entry is delayed after recessions, contributing to slow economic recoveries that we tend to observe empirically. The novel mechanism is that incumbents surviving a recession are more likely to be strong.

Proposition 3 (Path Dependence). *Fix $Z_{0-} \in \mathbb{R}$ and $X_0 \in (L(Z_{0-}), U(Z_{0-}))$. The entrant's equilibrium strategy has the form $\tau := \inf\{t > 0 : X_t \geq \hat{U}(M_t)\}$, where $M_t := \min_{0\leq s\leq t} X_s$, and where $\hat{U}$ is decreasing and nonconstant.*

Finally, the learning effect postpones entry, conditional on no revelation. Absent the prospect of learning, the entrant would optimally enter at the naive threshold U_0; note that this threshold incorporates the standard option value of waiting for conditions to improve; see, for example, Dixit and Pindyck (1994). But, in equilibrium, the prospect of learning implies that the equilibrium threshold U exceeds the naive threshold U_0: If the market falls, the entrant may learn more about the incumbent's cost type and make a more knowledgeable decision in the future. Consequently, the entrant postpones the entry decision and requires higher expected profits to enter.

Proposition 4 (Learning and Entry). *Fix $Z_{0-} \in \mathbb{R}$ and $X_0 \in (L(Z_{0-}), U(Z_{0-}))$. Suppose the incumbent is type s, or is type w and never reveals. Let $\tau_0 = \inf\{t > 0 : X_t \geq U_0(Z_{0-})\}$. Then, $\tau > \tau_0$.*

4. Discussion

In this section, we discuss various assumptions in the model, and we discuss how the model can be modified to study predatory pricing and how it extends to other forms of entry deterrence besides strategic pricing.

Market size: We assume that the market size follows a geometric Brownian motion. This assumption has two desirable features: (i) that the market experiences shocks both upward and downward, and (ii) there is a long-term industry growth trend (which can be positive or negative). The choice of geometric, rather than arithmetic, Brownian motion ensures that the market size remains positive. An alternative model could use a mean-reverting process, as in Dixit and Pindyck (1994, p. 161). We conjecture that our main qualitative results would survive.

Binary types: The assumption of binary types is mainly for simplicity, though it is natural that incumbents in a market might differ in discrete ways (e.g., an incumbent either has a particular technological capability or it does not). This assumption allows the entrant's beliefs to be summarized by a single-dimensional state variable, the probability that the incumbent is strong. However, qualitative features of our equilibrium would persist in a richer model with, say, a continuum of incumbent types. In any case, (i) the entrant would want to enter when the market is large relative to his beliefs about the incumbent, and (ii) weaker incumbent types would be tempted to reveal themselves when the market is small. One advantage of a continuum of incumbent types would be that, if the type distribution is atomless, equilibrium might not require mixing. However, the exact form of equilibrium would depend on more complicated conditions on parameter values.

Binary actions: In our model in Section 2, the (weak) incumbent faces a binary choice, signal, or reveal, which is a reduced form for choosing the strong incumbent's monopoly price or her own. We conjecture that the equilibrium we construct would remain an equilibrium with more actions available in each period corresponding to other prices—any deviation from the strong incumbent's monopoly price reveals the incumbent to be weak, but she maximizes her pre-entry flow profits by choosing her own monopoly price. On the other hand, enlarging the incumbent's strategy space would add significant complication to the definition of a strategy, which must allow for dynamic, stakes-dependent randomization.

Known entrant payoffs: Our assumption that the entrant's costs are common knowledge is natural in case the entrant is a well-known entity, and this assumption reduces the dimensionality of the problem. If, instead, the entrant also had private information, such as its marginal costs or its entry cost, the entrant's timing of entry would depend on its own private type, in addition to its belief about the incumbent and the current stakes. Private information on the entrant's side is explored in the model of Kolb (2015), where dynamics are driven by exogenous news about the entrant, rather than stochastic market conditions, and where the incumbent does not have private information. If one introduced private entrant types to the current model, any difference in entry timing across types would lead to learning by the incumbent at some states, and the incumbent's decision whether to continue its attempt to deter entry would then depend on its current belief about the entrant. We would, nonetheless, expect that under some specifications, pooling regions would remain, in which only stakes fluctuate, with learning about the entrant at high stakes and learning about the incumbent at low stakes.

Uninformative stakes: Our equilibrium features nontrivial belief dynamics, despite the fact that the only exogenous process built into the model is the uninformative stakes process. In fact, the absence of exogenous information arrival makes our model tractable. Specifically, value functions are relatively easy to characterize: Because beliefs do not update when the stakes are away from their historical minimum, value functions between the thresholds U and L satisfy well-known ODEs with respect to stakes, and they take the form of (generalized) polynomial functions of stakes, with coefficients that are functions of beliefs. Consequently, our construction reduces to a single ODE for the threshold U. Nonetheless, in some environments, exogenous information about the incumbent could arrive publicly over time. This could be modeled with an additional diffusion process with type-dependent drift, or, alternatively, by allowing the drift of our stakes process to be type-dependent. In either case, we would expect an equilibrium similar to ours to emerge, with entry at high stakes and (random) exit at low stakes. However, the exogenous information flow would complicate the boundary conditions at the thresholds U and L and would lead to a partial differential equation for value functions between these thresholds. Establishing existence and uniqueness in such an environment would therefore be a more demanding task, and we leave it for future work.

Reputation-independent flow payoffs: We also conjecture that the main features of our solution, which are entry at an upper threshold U that is increasing in reputation and mixing over exit at a lower threshold L that is decreasing in reputation, would be robust to some dependence of the incumbent's flow payoffs on the posterior belief about her type. However, the way in which flow payoffs prerevelation (S) and postrevelation (M^w) are differentially affected by reputation would be important. If the gap between M^w and S decreases in z, then we would expect our results to be reinforced: At high reputations, not only is the entry threshold higher, but the benefit of raising prices is smaller. However, if the gap between M^w and S is increasing in z, these two effects go in opposite directions, complicating the analysis.

4.1. Predatory Pricing

Our model can be modified to study predatory pricing, where one firm wants to induce the other to exit the market. As before, consider a game between two players, where Player 1 has a privately known type, strong or weak, depending on its marginal costs. The market has stochastically evolving demand X. In contrast to the limit pricing application, Player 2 is already present in the market, but can irreversibly exit at any time and receive an outside option of $Q > 0$ (Q could be interpreted as the scrap value of Player 2's assets). The predatory pricing game can be formulated in the Cournot market, as described in Section 3.5. Suppose the strong type of Player 1 is a natural monopolist and will obtain a flow proportional to M^s at all times. Player 1 of the weak type can imitate the behavior of the strong type and receive a flow proportional to S. Player 2 does not make a flow profit against the strong type, or against a weak type mimicking a strong type, and is therefore tempted to exit the market to earn the terminal payoff Q. In case Player 2 exits, the weak type of Player 1 receives higher flow $M^w > S$. Otherwise, if Player 1 is weak and reveals this before Player 2 exits, then the players compete in Cournot fashion and earn perpetual flow payoffs D_1^w and D_2^w, respectively. The most interesting case is when $S < D_1^w$, so that Player 1 must trade off a short-run incentive to reveal with a long-run incentive to force Player 2 to exit.

We conjecture that there exists an equilibrium that is nearly a mirror image of the equilibrium in the limit pricing game. A low enough market is unattractive compared with the outside option for Player 2, and, thus, Player 2 should exit at a low level of X. For Player 1 of the weak type, a large enough market offers room for a profitable duopoly vis à vis the cost of predatory pricing and the remote prospect of gaining a monopoly. Thus, the weak type has incentives to use predatory pricing for lower levels of X and randomize over revealing its type at a high level of X.

4.2. Earnings Management and Entry Deterrence

The economic forces present in the reduced-form signaling game of Section 2 can extend to other forms of entry deterrence besides strategic pricing. Consider, for example, an industry in which potential entrants observe an incumbent's earnings, but not its product pricing. The incumbent can manipulate earnings by changing accounting methods (accrual-based earnings management; see Dechow et al. 1995) or by altering real activities, such as production levels or R&D expenditures (real earnings management; see Roychowdhury 2006). The incumbent can then use earnings to signal its low costs and discourage entry. Specifically, a high-cost incumbent (Player 1, type w) can engage in earnings management to try to convince an entrant (Player 2) that it is a low-cost type. Dou et al. (2018) present evidence of such entry-deterrence behavior in the banking sector. Earnings management, either real earnings management or accrual-based earnings management, is costly for the incumbent (Zang 2012). It follows that the payoff of the high-cost incumbent engaging in earnings management to deter entry is lower than without earnings management, but is higher than the payoff after entry—consistent with the payoff structure in Table 1. The high-cost incumbent's willingness to engage in earnings management, and the entrant's decision to enter, will depend on the attractiveness of the market ("stake"), as in our baseline model.

5. Conclusion

We have presented a model of entry deterrence through strategic pricing in a stochastic environment and showed that such a setup generates novel strategic interactions between the incumbent and entrant. For a given belief about the type of the incumbent, the entrant has incentives to enter when the stakes become sufficiently high relative to the reputation of the incumbent. On the other hand, the incumbent has incentives to stop signaling at sufficiently low stakes, and, in equilibrium, the weak type separates herself incrementally through randomization. Despite the fact that our model does not include any exogenous information arrival about the incumbent, our model endogenously produces nontrivial belief dynamics, encoded in the running minimum of stakes.

The dependence on the minimum process brings a path-dependent market structure in equilibrium. The path dependence has a prescriptive implication for entrants: Their entry strategy should be based on the current value of the market demand and on the posterior expectation of the strength of the incumbent, which can be inferred from the past evolution of the demand. The model also implies that observable price dynamics may reveal entry-deterring pricing practices. We find that a stochastic environment with incomplete information creates pre-entry learning opportunities about the incumbent's type and delays entry.

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Appendix A. Proof of Lemma 1

First, note that at $\tilde{U} = \tilde{u}_T$, the equation $G(\tilde{U}, \Delta) = 0$ reduces to $(1-\beta_2)e^{(\beta_1-1)\Delta} + (\beta_1 - 1)e^{(\beta_2-1)\Delta} - \overline{\beta} = 0$, which has a unique solution $\Delta = 0$, and, thus, $f(\tilde{u}_T) = 0$. Next, note that G is strictly decreasing in $\tilde{U}$, so for all $\tilde{U} > \tilde{u}_T$, $G(\tilde{U}, 0) < 0$, while $\lim_{\Delta\to\infty} G(\tilde{U}, \Delta) = \infty$. Thus, for all $\tilde{U} > \tilde{u}_T$, there exists a unique $f(\tilde{U})$ such that $G(\tilde{U}, f(\tilde{U})) = 0$. Further, we have $G_\Delta(\tilde{U}, \Delta) = \frac{(\beta_1-1)(1-\beta_2)}{\overline{\beta}} \frac{M^w - S}{r-\mu}(e^{(\beta_1-1)\Delta} - e^{(\beta_2-1)\Delta})$, which is strictly positive if and only if $\Delta > 0$. By the implicit function theorem,

f is continuously differentiable, and

$$f'(\tilde{U}) = \frac{-G_{\tilde{U}}(\tilde{U}, f(\tilde{U}))}{G_\Delta(\tilde{U}, f(\tilde{U}))} = -\frac{(\beta_1 - 1)A_T e^{(\beta_1-1)\tilde{U}}}{\frac{(\beta_1-1)(1-\beta_2)}{\overline{\beta}}\frac{M^w - S}{r-\mu}\left(e^{(\beta_1-1)f(\tilde{U})} - e^{(\beta_2-1)f(\tilde{U})}\right)}.$$

Substituting from (19) in the numerator and simplifying, we have

$$f'(\tilde{U}) = \frac{(1-\beta_2)(M^w - S)e^{\overline{\beta}f(\tilde{U})} + (\beta_1 - 1)(M^w - S) + \overline{\beta}(S - D_1^w)e^{(1-\beta_2)f(\tilde{U})}}{(1-\beta_2)(M^w - S)(e^{\overline{\beta}f(\tilde{U})} - 1)}, \tag{A.1}$$

And, by inspection, $f'(\tilde{U}) > 1$. Thus, as $\tilde{U} \to \infty, f(\tilde{U}) \to \infty$. Then, because the numerator and denominator of (A.1) have the same leading term, $\lim_{\tilde{U}\to\infty} f'(\tilde{U}) = 1$.

Appendix B. Supporting Expressions for Sections 3.3 and 3.4

Given $\tilde{U}(z)$, the general solutions to (6), (20), and (21) are

$$B_1(z) = \frac{1}{\overline{\beta}}\left(\frac{D_2^w(1-\beta_2)}{(1+e^z)(r-\mu)}e^{(1-\beta_1)\tilde{U}(z)} + K\beta_2 e^{-\beta_1\tilde{U}(z)}\right), \tag{B.1}$$

$$B_2(z) = \frac{1}{\overline{\beta}}\left(\frac{D_2^w(\beta_1-1)}{(1+e^z)(r-\mu)}e^{(1-\beta_2)\tilde{U}(z)} - K\beta_1 e^{-\beta_2\tilde{U}(z)}\right). \tag{B.2}$$

As described in Section 3.3, we obtain an ordinary differential equation for $\tilde{U}$:

$$\tilde{U}' = \frac{[D_2^w(\beta_1 - 1)e^{\overline{\beta}(\tilde{U}-\tilde{L})+\tilde{U}} - \beta_1 K(r-\mu)e^{\overline{\beta}(\tilde{U}-\tilde{L})} - \overline{\beta}B_T(r-\mu)e^{\beta_1\tilde{U}} + D_2^w(1-\beta_2)e^{\tilde{U}} + \beta_2 K(r-\mu)]}{(e^{\overline{\beta}(\tilde{U}-\tilde{L})} - 1)(D_2^w(\beta_1-1)(1-\beta_2)e^{\tilde{U}} + \beta_1\beta_2 K(r-\mu)(1+e^z))}. \tag{B.3}$$

Next, we derive the function $\tilde{U}^+(z)$ defined in Section 3.4. Player 2's value function is of the form $V^+(x,z) = B_1^+(z)e^{\beta_1 x} + B_2^+(z)e^{\beta_2 x}$ where, using (B.1) and (B.2), we have

$$B_1^+(z) = \frac{1}{\overline{\beta}}\left(\frac{D_2^w(1-\beta_2)}{(1+e^z)(r-\mu)}e^{(1-\beta_1)\tilde{U}^+(z)} + K\beta_2 e^{-\beta_1\tilde{U}^+(z)}\right), \tag{B.4}$$

$$B_2^+(z) = \frac{1}{\overline{\beta}}\left(\frac{D_2^w(\beta_1-1)}{(1+e^z)(r-\mu)}e^{(1-\beta_2)\tilde{U}^+(z)} - K\beta_1 e^{-\beta_2\tilde{U}^+(z)}\right). \tag{B.5}$$

Now, $\tilde{U}^+(z)$ is pinned down by the following boundary condition based on full separation of Player 1 types at $\tilde{u}_T$:

$$\begin{aligned} &V^+(\tilde{u}_T, z) = \frac{1}{1+e^z}V_T(\tilde{u}_T) = \frac{1}{1+e^z}\left[\frac{D_2^W e^{\tilde{u}_T}}{r-\mu} - K\right] \\ &\iff B_1^+(z)e^{\beta_1\tilde{u}_T} + B_2^+(z)e^{\beta_2\tilde{u}_T} - \frac{1}{1+e^z}\left[\frac{D_2^w e^{\tilde{u}_T}}{r-\mu} - K\right] = 0. \end{aligned} \tag{B.6}$$

To see that $\tilde{U}^+(z)$ is well defined, let $J(\tilde{U})$ denote the left side of (B.6), which depends on $\tilde{U}$ through B_1^+ and B_2^+. It is easy to verify that $J(\tilde{u}_T) < 0$ and $\lim_{\tilde{U}\to+\infty} J(\tilde{U}) = +\infty$. Further, we have

$$\begin{aligned} J_{\tilde{U}}(\tilde{U}, z) &= \frac{1}{\overline{\beta}}(e^{-\beta_2(\tilde{U}-\tilde{u}_T)} - e^{-\beta_1(\tilde{U}-\tilde{u}_T)})\left[\frac{(1-\beta_2)(\beta_1-1)D_2^w}{(1+e^z)(r-\mu)} + K\beta_1\beta_2\right] \\ &\geq \frac{1}{\overline{\beta}}\left(e^{-\beta_2(\tilde{U}-\tilde{u}_T)} - e^{-\beta_1(\tilde{U}-\tilde{u}_T)}\right)\beta_1 K > 0. \end{aligned}$$

Thus, $J(\tilde{U}, z)$ is strictly increasing in $\tilde{U}$ for $\tilde{U} > \tilde{u}_T$, and there is a unique root $\tilde{U}^+(z) > \tilde{u}_T$.

Appendix C. Proofs for Section 3.4

Lemmas 2–4 establish that there exists a unique candidate for a (U, L) equilibrium. The proofs of Lemmas 2 and 4 are algebraically cumbersome, so we provide them in the electronic companion. Sections 3.1–3.3 characterize the candidate value functions associated with (U, L).

Proof of Lemma 3. For all z, a lower bound on $V(x, z)$ is Player 2's value when the weak type's strategy is to never reveal, and, thus, it is easy to see that $\tilde{U}(z) \geq \tilde{U}_0(z)$. It follows that $\tilde{U}(z) > \tilde{u}_T$ for all $z > -\infty$, and, thus, Lemma 1 implies $\tilde{L}(z) = \tilde{U}(z) - f(\tilde{U}(z)) < \tilde{u}_T - f(\tilde{u}_T) = \tilde{u}_T$ for all $z > -\infty$. Hence, an upper bound on $V(x, z)$ is Player 2's value when the weak type's strategy is to reveal with certainty when stakes fall below $\tilde{u}_T$; the optimal entry threshold against such a strategy is $\tilde{U}^+(z)$, and, thus, $\tilde{U}^+(z) \geq \tilde{U}(z)$. □

To prove Theorem 1, all that remains is to verify optimality for both players; the other properties follow from the construction.

The following lemma aids in our verification. The proof, contained in the electronic companion, is mainly algebraic and leverages the value-matching and smooth-pasting conditions from Section 3.

Define $U(+\infty) = +\infty$, $U(-\infty) = u_T$, $L(+\infty) = \underline{L}$, and $L(-\infty) = u_T$. It is clear from our construction that for all $z \in \bar{\mathbb{R}}$, $x \longmapsto F(x,z)$ is C^2 on $(L(z), U(z))$ and $x \longmapsto V(x,z)$ is C^1 on $(L(z), +\infty)$ and C^2 on $(L(z), +\infty) \setminus \{U(z)\}$.

Lemma C.1. *The players' candidate value functions F and V have the following properties*:

i. $F(x,z) \geq F_T(x)$ *on* $\{(x,z) \in \mathbb{R}_{>0} \times \bar{\mathbb{R}} : x \leq U(z)\}$, *with strict inequality on* $\{(x,z) \in \mathbb{R}_{>0} \times \bar{\mathbb{R}} : x \in (L(z), U(z))\}$.

ii. $rF(x, +\infty) > \mu x F_x(x, +\infty) + \frac{\sigma^2 x^2}{2}F_{xx}(x, +\infty) + Sx$ *on* $(0, L(+\infty))$.

iii. $V(x,z) \geq \frac{D_2^w}{(1+e^z)(r-\mu)}x - K$ *on* $\{(x,z) \in \mathbb{R}_{>0} \times [-\infty, +\infty) : x \geq L(z)\} \cup \{(x, +\infty) : x \in \mathbb{R}_{>0}\}$.

iv. $rV(x,z) > \mu x V_x(x,z) + \frac{\sigma^2 x^2}{2}V_{xx}(x,z)$ *on* $\{(x,z) \in \mathbb{R}_{>0} \times \bar{\mathbb{R}} : x > U(z)\}$.

For the verification, it is useful to change variables for beliefs back from z to π and write $F(x,\pi)$ and $V(x,\pi)$ for the value functions and $U(\pi)$ and $L(\pi)$ for the players' thresholds.

We begin by verifying optimality for Player 2. Given $(X_0, \pi_0) = (x, \pi)$, with probability one, for all $t \geq 0$, $X_t \geq L(\pi_t)$ or $\pi_t = 1$ due to the belief dynamics in the candidate equilibrium. Thus, by Lemma C.1, part (iii),

$$e^{-rt}\left((1-\pi_t)\frac{D_2^w}{r-\mu}X_t - K\right) \leq e^{-rt}V(X_t, \pi_t), \tag{C.1}$$

and by construction, there is equality when $X_t \geq U(\pi_t)$.

Now, define $m_t = \int_0^t e^{-rs}\sigma X_s V_x(X_s, \pi_s)dW_s$ and $y_t = \int_0^t e^{-rs} V_\pi(X_{s-}, \pi_{s-})d\pi_s$. For all $N > 0$, define $\nu_N = N \wedge \inf\{t \geq 0 : |m_t| \geq N \text{ or } |y_t| \geq N\}$, so that $m_{t\wedge\nu_N}$ and $y_{t\wedge\nu_N}$ are martingales. Decompose π_t into a part with jumps $\pi_t^\Delta := \pi_t - \pi_{t-}$

and a continuous part $\pi_t^C := \pi_t - \sum_{0<s\le t}\pi_s^\Delta$. By Itô's lemma,

$$\begin{aligned} e^{-r(\tau\wedge\nu_N)}V(X_{\tau\wedge\nu_N},\pi_{\tau\wedge\nu_N}) &= V(X_0,\pi_0)+m_{\tau\wedge\nu_N}\\ &+\int_0^{\tau\wedge\nu_N} e^{-rs}\left(-rV(X_s,\pi_s)+\mu X_s V_x(X_s,\pi_s)+\frac{\sigma^2X_s^2}{2}V_{xx}(X_s,\pi_s)\right)ds\\ &+\int_0^{\tau\wedge\nu_N} e^{-rs}V_\pi(X_s,\pi_s)d\pi_s^C+\sum_{0\le s\le\tau\wedge\nu_N} e^{-rs}(V(X_s,\pi_s)-V(X_{s-},\pi_{s-})), \end{aligned} \tag{C.2}$$

where $V_{xx}(x,\pi)$ can be defined arbitrarily at the curve $\{(U(\pi),\pi):\pi\in[0,1)\}$.

Next, we argue that the third term on the right-hand side of (C.2) is nonpositive. Note that for all $s\in[0,t]$, except on a set of measure zero, we have $X_s\in(L(\pi_s),\infty)\setminus\{U(\pi_s)\}$, and for such s, the integrand vanishes for $X_s\in(L(\pi_s,U(\pi_s))$ by construction and is strictly negative for $X_s>U(\pi_s)$ by Lemma C.1, part (iv).

Taking expectations on both sides of the resulting inequality, and using that $\mathbb{E}[m_{\tau\wedge\nu_N}]=0$ by the martingale property,

$$\begin{aligned} \mathbb{E}[e^{-r(\tau\wedge\nu_N)}V(X_{\tau\wedge\nu_N},\pi_{\tau\wedge\nu_N})] &\le V(X_0,\pi_0)\\ &+\mathbb{E}\int_0^{\tau\wedge\nu_N} e^{-rs}V_\pi(X_s,\pi_s)\,d\pi_s^C+\\ \mathbb{E}\Bigg[\sum_{0<s\le\tau\wedge\nu_N} &e^{-rs}(V(X_s,\pi_s)-V(X_{s-},\pi_{s-}))\Bigg]. \end{aligned} \tag{C.3}$$

To simplify the last two terms of (C.3), note that the difference $V(X_s,\pi_s)-V(X_{s-},\pi_{s-})$ is nonzero only when $\pi_s\neq\pi_{s-}$—that is, when $X_{s-}=L(\pi_{s-})$ and Player 1 reveals as w, in which case $\pi_s=0$. Now, after a change of variables, (22) is equivalent to $\pi V_\pi(L(\pi),\pi)=V(L(\pi),\pi)-V_T(L(\pi))$. It follows that the last term in (C.3) equals

$$\begin{aligned} &\mathbb{E}\Bigg[-\sum_{0<s\le\tau\wedge\nu_N} e^{-rs}\pi_{s-}V_\pi(X_{s-},\pi_{s-})\Bigg]\\ &=\mathbb{E}\Bigg[\sum_{0<s\le\tau\wedge\nu_N} e^{-rs}V_\pi(X_{s-},\pi_{s-})\pi_s^\Delta\Bigg]. \end{aligned}$$

Hence, the last two terms in (C.3) sum to $\mathbb{E}[y^{\tau\wedge\nu_N}]=0$ by the martingale property, reducing (C.3) to

$$\mathbb{E}[e^{-r(\tau\wedge\nu_N)}V(X_{\tau\wedge\nu_N},\pi_{\tau\wedge\nu_N})]\le V(X_0,\pi_0).$$

Taking $N\to\infty$ and applying Fatou's lemma yields $\mathbb{E}[e^{-r\tau}V(X_\tau,\pi_\tau)]\le V(X_0,\pi_0)$. Finally, evaluating (C.1) at τ and taking expectations, we get the desired inequality

$$\begin{aligned} \mathbb{E}\left[e^{-r\tau}\left((1-\pi_\tau)\frac{D_2^w}{r-\mu}X_\tau-K\right)\right] &\le \mathbb{E}[e^{-r\tau}V(X_\tau,\pi_\tau)]\\ &\le V(X_0,\pi_0), \end{aligned} \tag{C.4}$$

where the left side is Player 2's expected payoff (3) from an arbitrary strategy. Moreover, by construction, equality holds when Player 2 plays the candidate equilibrium strategy $\tau=\inf\{t\ge0: X_t\ge U(\pi_t)\}$. Thus, this strategy is optimal for Player 2.

Next, we verify optimality for Player 1, type w, given that Player 2 enters at $\tau=\inf\{t\ge0:X_t\ge U(\pi_t)\}$. Let ρ be an arbitrary stopping time at which type w reveals. By Itô's lemma,

$$\begin{aligned} e^{-r(\rho\wedge\tau)}F(X_{\rho\wedge\tau},\pi_{\rho\wedge\tau}) &= F(X_0,\pi_{0-})+\int_0^{\rho\wedge\tau} e^{-rs}\sigma X_sF_x(X_s,\pi_s)\,dW_s\\ &+\int_0^{\rho\wedge\tau} e^{-rs}\left(-rF+\mu X_sF_x+\frac{\sigma^2X_s^2}{2}F_{xx}\right)ds\\ &+\int_0^{\rho\wedge\tau} e^{-rs}F_\pi d\pi_s^C+\mathbb{1}\{\rho\le\tau\}e^{-r\rho}(F(X_\rho,0)-F(X_{\rho-},\pi_{\rho-})). \end{aligned} \tag{C.5}$$

Note that although π jumps at time 0 if the game starts below L, there is no associated jump in type w's continuation value, so there is no need to include such a term.

The penultimate term vanishes because, prior to revelation, π_s^C only increases when $X_s=L(\pi_s)$, and $F_\pi=0$ along this curve by (17).

We argue that the third term on the right-hand side of (C.5) is at most $\int_0^{\rho\wedge\tau}-e^{-rs}SX_s\,ds$, and it is zero for all ρ in the support of Player 1's equilibrium strategy. For all $s\in[0,\rho\wedge\tau]$ outside a set of measure zero, the belief dynamics imply that either (a) $X_s\in(L(\pi_s),U(\pi_s))$ (including the possibility that $\pi_s=1$) or (b) $\pi_s=1$ and $X_s<L(1)=\underline{L}$. In case (a), the integrand at hand equals $-e^{-rs}SX_s$ by construction. In case (b), the integrand is strictly less than $-e^{-rs}SX_s$ by Lemma C.1, part (ii). This establishes weak inequality. Moreover, for ρ in the support of Player 1's equilibrium strategy, we have $\rho\le\inf\{t>0:X_t\le\underline{L}\}$, so case (b) has measure zero, and the integral is zero.

In the last term of (C.5), note that $F(X_\rho,0)-F(X_{\rho-},\pi_{\rho-})=F_T(X_\rho)-F(X_{\rho-},\pi_{\rho-})\le0$ by Lemma C.1, part (i). We further argue that for all ρ in the support of type w's strategy, $F(X_\rho,0)-F(X_{\rho-},\pi_{\rho-})=0$. To see this, recall that for every such ρ, we have $\rho=\inf\{t>0:X_t=x\}$ for some $x\in[\underline{L},L(\pi_{0-})]$. It follows that $X_{\rho-}=L(\pi_{\rho-})$, and, thus, by the value-matching Conditions (15) and $F(\underline{L},1)=F_T(\underline{L})$, $F(X_{\rho-},\pi_{\rho-})=F_T(X_\rho)=F(X_\rho,0)$.

Redefine $m_t=\int_0^t e^{-rs}\sigma X_sF_x(X_s,\pi_s)\,dW_s$, and for all $N>0$, redefine $\nu_N=N\wedge\inf\{t\ge0:|m_t|\ge N\}$, so that $m_{t\wedge\nu_N}$ is a martingale, and, thus, $\mathbb{E}[m_{\rho\wedge\tau\wedge\nu_N}]=0$. Replacing ρ with $\rho\wedge\nu_N$ in (C.5), taking expectations, and incorporating the facts established above,

$$\begin{aligned} F(X_0,\pi_{0-}) &\ge \mathbb{E}\left[\int_0^{\rho\wedge\tau\wedge\nu_N} e^{-rs}SX_s ds\right]\\ &+\mathbb{E}\left[e^{-r(\rho\wedge\tau\wedge\nu_N)}F(X_{\rho\wedge\tau\wedge\nu_N},\pi_{\rho\wedge\tau\wedge\nu_N})\right], \end{aligned}$$

whereby taking $N\to\infty$ and invoking Fatou's lemma yields

$$F(X_0,\pi_{0-})\ge\mathbb{E}\left[\int_0^{\rho\wedge\tau} e^{-rs}SX_s ds\right]+\mathbb{E}\left[e^{-r(\rho\wedge\tau)}F(X_{\rho\wedge\tau},\pi_{\rho\wedge\tau})\right].$$

Finally, note that when $\tau<\rho$, we have $F(X_{\rho\wedge\tau},\pi_{\rho\wedge\tau})=D_1^wX_\tau/(r-\mu)$; and when $\rho\le\tau$, $F(X_{\rho\wedge\tau},\pi_{\rho\wedge\tau})=F_T(X_\rho)$. It follows that

$$\begin{aligned} &F(X_0,\pi_{0-})\ge\\ &\mathbb{E}\left[\int_0^{\rho\wedge\tau} e^{-rs}SX_s ds+\mathbb{1}\{\tau<\rho\}e^{-r\tau}\frac{D_1^w}{r-\mu}X_\tau+\mathbb{1}\{\rho\le\tau\}e^{-r\rho}F_T(X_\rho)\right], \end{aligned}$$

where the right-hand side is precisely (2). Moreover, we have equality when ρ is in the support of w's candidate equilibrium strategy. Thus, this strategy is optimal for type w.

Appendix D. Proofs for Section 3.6

Proof of Proposition 2. Let P^s and $P^w > P^s$ denote the prices induced by type s and w, respectively, as monopolists (see Section 3.5). When Player 1 is type w, the price is P^s until w reveals, and then it jumps to P^w. Because $X_0 \in (L(Z_{0-}), U(Z_{0-}))$, $P_0 = P^s$. Now, fix $t > 0$. With positive probability, Player 1 is type w, and X_t hits $\underline{L}$ before $U(Z_{0-})$; conditional on this event, type w reveals almost surely before time t, and Player 2 remains out of the market at t, so that $P_t = P^w > P_0$. But because $\underline{L} \le L(Z_{0-}) < X_0$, we have $X_t < X_0$ and, thus, $Y_t < Y_0$, where $X = Y^\gamma$. □

Proof of Proposition 3. Let

$$\hat{U}(m) = \begin{cases} U(Z_{0-}) & \text{if } m \ge L(Z_{0-}) \\ U(L^{-1}(m)) & \text{if } m \in (\underline{L}, L(Z_{0-}))\,. \\ +\infty & \text{if } m \le \underline{L} \end{cases} \tag{D.1}$$

Clearly, $\hat{U}$ is nonconstant, and because U is increasing and L^{-1} is decreasing, $\hat{U}$ is decreasing. By construction, $\hat{U}(M_t) = U(Z_t)$ for all $t \ge 0$. Because $\tau = \inf\{t > 0 : X_t \ge U(Z_t)\}$, the proof is complete. □

Proof of Proposition 4. Recall that $\tau = \inf\{t > 0 : X_t \ge U(Z_t)\}$, and observe that when Player 1 is type s or is type w and never reveals, Z_t is nondecreasing. Hence, $\tau \ge \inf\{t > 0 : X_t \ge U(Z_{0-})\} > \inf\{t > 0 : X_t \ge U_0(Z_{0-})\}$, where the last inequality follows from the inequality $U(Z_{0-}) > U_0(Z_{0-})$. □

Endnotes

[1] Décamps and Mariotti (2004) study a leader-follower investment game with learning externalities, in which players have symmetric uncertainty about the value of investment and private information about their investment costs, but the strategic interaction there is a war of attrition, and, due to the learning technology, the historical maximum is redundant.

[2] For discussion of this and other model features, see Section 4.

[3] One interpretation of this setup is that the strong type of Player 1 always chooses her optimal monopoly price, and the weak type chooses how long to mimic the strong type before giving up and reverting to her own, higher monopoly price.

[4] Because Player 1's decision to reveal is only payoff-relevant prior to entry by Player 2, it is not necessary for Player 1's decision to condition on the time of entry.

[5] It is useful to work with the log transform of the market size. As a convention, we use a tilde to denote the natural log of any market size quantity: For example, $\tilde{x} = \ln(x)$, $\tilde{U}(z) = \ln(U(z))$ and $\tilde{L}(z) = \ln(L(z))$.

[6] In models of reputation driven by exogenous news and signaling where similar "reflected" processes arise, the reflected process is usually the reputation process itself and is, thus, one-dimensional (Daley and Green 2012, Kolb 2019). Here, in contrast, neither the market size process (which is exogenous) nor the reputation process (which is monotonic, conditional on no revelation) alone is reflected, but, rather, the pair.

[7] In particular, they are the roots of the characteristic equation $\chi(\beta) := \frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0$: $\beta_1, \beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$. Because $\chi(1) = \mu - r < 0$ and $\chi(0) = -r < 0$, we have $\beta_1 > 1$ and $\beta_2 < 0$.

[8] See Peskir (1998, p. 1620).

[9] What is also true, but cannot be shown in a figure of reasonable scale, is that all the other solutions to (B.3) shown in Figure 2(b) converge asymptotically to $\tilde{U}_0$ as $z \to \infty$. For this reason, asymptotic convergence to $\tilde{U}_0$ cannot be used to pin down a solution.

[10] This assumption allows us to pin down discounted continuation payoffs to serve as termination payoffs for the pre-entry game, which is the focus of our analysis.

[11] Specifically, $M^\theta(\tilde{C}) = \frac{\gamma\tilde{C} - (\gamma-1)C_1^\theta}{\gamma-1}\left(\frac{\gamma\tilde{C}}{\gamma-1}\right)^{-\gamma}$; D_1^θ equals $\frac{[\gamma C_2 - (\gamma-1)C_1^\theta]^2}{(2\gamma-1)(C_1^\theta + C_2)}\left[\frac{\gamma(C_1^\theta + C_2)}{2\gamma-1}\right]^{-\gamma}$ if $C_1^\theta \in [\frac{\gamma-1}{\gamma}C_2, \frac{\gamma}{\gamma-1}C_2]$, M^θ if $C_1^\theta < \frac{\gamma-1}{\gamma}C_2$, and zero if $C_1^\theta > \frac{\gamma}{\gamma-1}C_2$; and D_2^θ equals $\frac{[\gamma C_1^\theta - (\gamma-1)C_2]^2}{(2\gamma-1)(C_1^\theta + C_2)}\left[\frac{\gamma(C_1^\theta + C_2)}{2\gamma-1}\right]^{-\gamma}$ if $C_2 \in [\frac{\gamma-1}{\gamma}C_1^\theta, \frac{\gamma}{\gamma-1}C_1^\theta]$, $\frac{1}{\gamma}\left(\frac{\gamma C_2}{\gamma-1}\right)^{1-\gamma}$ if $C_2 < \frac{\gamma-1}{\gamma}C_1^\theta$, and zero if $C_2 > \frac{\gamma}{\gamma-1}C_1^\theta$.

[12] There is a nonempty set of values for $(\gamma, C_1^s, C_1^w, C_2)$ satisfying our assumptions.

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Sebastian Gryglewicz is a professor of Finance at Erasmus University Rotterdam. His research interests include corporate finance, financial intermediation, and dynamic contracts.

Aaron Kolb is an associate professor of Business Economics and Public Policy at the Indiana University Kelley School of Business. His research interests include dynamic stochastic games, dynamic contracts, and strategic transmission of private information.