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A theory of corporate financial decisions with liquidity and solvency concerns ☆

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ABSTRACT

This paper studies the impact of both liquidity and solvency concerns on corporate finance. I present a tractable model of a firm that optimally chooses capital structure, cash holdings, dividends, and default while facing cash flows with long-term uncertainty and short-term liquidity shocks. The model explains how changes in solvency affect liquidity and also how liquidity concerns affect solvency via capital structure choice. These interactions result in a dynamic cash policy in which cash reserves increase in profitability and are positively correlated with cash flows. The optimal dividend distributions implied by the model are smoothed relative to cash flows. I also find that liquidity concerns lead to a decrease of dispersion of credit spreads.

1. Introduction

Financial distress is recognized as a driving force behind many corporate decisions. At the same time, however, there is little understanding of the roles of and relations between corporate illiquidity and insolvency—the two sources of financial distress. In this paper, I argue that the interactions of liquidity and solvency can explain empirical patterns in cash and dividend policies, and also shed further light on capital structure choice, valuation, and credit spreads.

Corporate finance literature has long been interested in how firms that generate uncertain cash flows distribute dividends. Firms paying dividends tend to smooth distributions relative to earnings and, when in distress, they reduce dividends rather than omit them (Lintner, 1956; DeAngelo and DeAngelo, 1990; Brav, Graham, Harvey, and Michaely, 2005). Leary and Michaely (2008) show that dividend smoothing has been steadily increasing over the past decades, but the reasons for such a payout policy remain a puzzle. A related question is how much cash firms save out of high cash flows and disburse in periods of low cash flows. Corporate cash policies have been recently receiving increasing attention due to the vast and growing cash holdings of U.S. companies (see Bates, Kahle, and Stulz, 2009). Financially constrained firms appear to show a positive cash flow sensitivity of cash, that is, a propensity to save cash from positive cash flow shocks (Almeida,

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Campello, and Weisbach, 2004; Khurana, Martin, and Pereira, 2006; Sufi, 2009). However, the direction of the sensitivity and the reasoning behind it are unsettled (see Riddick and Whited, 2009).[1] The prevailing evidence indicates that corporate cash serves mainly as a buffer against adverse cash flow shocks.

To provide unifying insights into cash and dividend policies, this paper proposes a tractable dynamic model of corporate finance that integrates liquidity and solvency concerns and uncovers linkages between them. Consistent with the above empirical facts, my analysis reveals that firms that use cash to hedge liquidity shocks hold large amounts of cash, smooth dividends, and exhibit a positive cash flow sensitivity of cash. In essence, the model shows that persistent liquidity shocks affect solvency and that solvency levels, in turn, determine demand for corporate liquidity. For example, a negative cash flow surprise decreases solvency and, consequently, such a firm requires less cash. Because any excess cash is distributed, dividend payment is smoothed and the shock is absorbed by cash holdings.

To motivate the approach taken in this paper, I start with some elementary properties of liquidity and solvency risks. Corporate liquidity is a short-term characteristic that measures the ability of a firm to pay its obligations on time. Corporate solvency is the ability to cover debt obligations in the long run. Liquidity and solvency risks are closely related to cash flow uncertainty. Short-term shocks to cash flows, together with the availability of cash reserves, affect corporate liquidity. Uncertainty about average future profitability, together with financial leverage, generates solvency concerns.[2] These relations indicate that firms enter financial distress in two ways: a firm can become illiquid after a negative short-term cash flow or it can become insolvent if the expected rate of cash flows decreases sufficiently.

The defining characteristic of the model is that it recognizes that these two sources of cash flow shocks are *separate* but *interconnected*. If a firm generates negative liquidity surprises, that is, if cash flows persistently fall below their expected level, expectations about future cash flow are adjusted downwards. Conversely, a firm that persistently generates positive liquidity shocks must be, after all, more profitable than expected. In both situations, liquidity shocks accumulate to change expected firm value and thereby solvency.

To disentangle solvency and liquidity concerns, I model cash flows with two sources of uncertainty. The first is short-term liquidity uncertainty: at each time cash flow realizations may fall above or below expected cash flow due to a liquidity shock. The second source of uncertainty is long-term solvency uncertainty: the expected cash flow rate evolves over time. Liquidity and solvency are connected because short-term liquidity shocks affect the expected cash flow rate through Bayesian learning. More specifically, it is assumed that cash flows follow a Brownian motion with drift and that the drift parameter is not directly observable. The firm and investors observe noisy cash flows (subject to liquidity shocks) and learn about the drift (the average rate of cash flows). In this way, short-term liquidity shocks around average profitability are not only noise but also, if persistent, affect the assessment of solvency.

I embed this cash flow process in a model of dynamic corporate finance with financing constraints, endogenous capital structure, dividend policy, cash holdings, and default. In the model, the firm issues a combination of equity and debt to finance the required investment and initial cash. Corporate debt offers a tax advantage but also creates bankruptcy costs. The firm generates cash flows with two sources of uncertainty, pays debt coupons and taxes. At each time, positive net earnings can be either distributed as dividends or retained to increase cash holdings. Net losses and dividend payouts can be covered from cash reserves. The payout-retention policy maximizes equity value. If at any time the firm is unable to pay its obligations, it is illiquid. If firm value falls below debt value, the firm becomes insolvent and it defaults. The model uncovers several linkages between liquidity and solvency, and underscores their roles in cash, dividend, and leverage policies.

In the presence of financing constraints, a firm without sufficient cash reserves may become illiquid and be forced into default while still solvent. The model characterizes a level of cash, denoted by $\overline{C}$, that allows the firm to withstand liquidity shocks up to the point at which the equity holders endogenously trigger solvency default. The analysis shows that $\overline{C}$ evolves over time and increases with expected profitability. Intuitively, a more profitable firm is more solvent and thus, it has a greater continuation value that is to be saved. Such a firm is willing to withstand larger liquidity shocks before it is eventually deemed insolvent and so the required cash buffer is higher. In other words, higher solvency creates a higher demand for corporate liquidity.

I further show that it is optimal for a firm that maximizes equity value to retain all earnings if cash is below $\overline{C}$ and, subsequently, to pay out dividends that allow the firm to maintain cash at $\overline{C}$. The model predicts dividend distributions that are smooth in comparison with cash flows or earnings. The reason is that the target cash level $\overline{C}$ is not constant but increases and decreases with firm value. A positive earnings surprise provides some positive information about future cash flows, increases expected firm value, and thereby also the optimal level of $\overline{C}$. Therefore, not all but only a fraction of additional earnings will be paid out as a dividend. Conversely, if earnings are surprisingly low, firm value and cash level $\overline{C}$ decrease, so that dividends are complemented by released cash holdings.

[1] Riddick and Whited (2009) use an alternative empirical methodology to the one used by Almeida, Campello, and Weisbach (2004). The reasoning employed in the literature on cash flow sensitivity of cash relies on the assumption that cash is used mainly to fund future investments. However, both empirical and survey evidence in Opler, Pinkowitz, Stulz, and Williamson (1999) and Lins, Servaes, and Tufano (2010) shows that the main motive for corporate cash holdings is precautionary, but rather than to fund investments, cash is used to cushion adverse cash flow shocks.

[2] I use the terms "liquidity" and "short-term" interchangeably to describe liquidity-related shocks and risks. Similarly, "solvency," "long-term," and "profitability" are used interchangeably for solvency shocks and risks.

This mechanism simultaneously explains dividend smoothing and positive cash flow sensitivity of cash. The insensitivity of dividends and the sensitivity of cash to cash flow shocks must be closely related because, without external financing of dividends, payouts can be smooth only if cash reserves absorb cash flow shocks. This model is the first to show how this corporate policy arises from an equity-value-maximizing dividend policy through the interplay of liquidity and solvency.

Apart from implications for cash and dividend policies, the model has consequences also for capital structure. The model predicts that firms select their leverage to limit exposure to liquidity risk. In this way, liquidity has an impact on capital structure and thereby on solvency. This results in a novel trade-off in debt choice. Additional cash holdings are costly, so debt should be chosen such that required cash levels are low. If debt is very low, solvency is high and demand for cash is also high. On the other hand, if debt is very high, the liquidity pressure by debt coupon payments also lead to high required cash holdings. Consequently, an intermediate level of debt is optimal. As the changes in leverage translate ultimately into changes in debt credit spreads, the model predicts lower dispersion of credit spreads across firms than in the standard environment without liquidity concerns. Empirically, Eom, Helwege, and Huang (2004) show that existing models tend to predict credit spreads that are too high if observed spreads are relatively high, while predicted spreads are too low if observed spreads are relatively low. The recognition of liquidity concerns in this paper moves credit spreads in the empirically observed direction.

Further analysis indicates that short-term cash flow volatility and long-term uncertainty about profitability can have very different effects on financial variables. I find that cash holdings increase in volatility and decrease in profitability uncertainty. The first relation is in line with the explanation of Bates, Kahle, and Stulz (2009) for high cash levels among U.S. firms. The second prediction is novel and provides grounds for further empirical tests of determinants of cash holdings. Debt credit spreads also react differently to the two measures of risk, namely, they decrease in volatility and increase in profitability uncertainty. The two sources of uncertainty have different effects because, in essence, volatility is related to liquidity and profitability uncertainty to solvency concerns.

In the following section, I set up the model. Section 3 analyzes a benchmark case of a firm without financing constraints concerned only about solvency. Sections 4 and 5 present the main model with both liquidity and solvency concerns and discuss its implications. Section 4 analyzes optimal cash and dividend policies. Section 5 examines capital structure. In Section 6, I relate the paper to previous literature. Section 7 concludes.

2. Setup

2.1. Outline and timing

I consider financial decisions of a firm that generates uncertain cash flows. The firm selects its capital structure, cash holdings, dividend payout, and default policy. The model is set in continuous time with an infinite horizon; time is indexed as $t \in [0,\infty)$. It is assumed that management acts in the interest of equity holders, all investors are risk neutral and discount cash flows at a constant risk-free rate r.

The original equity holders are financially constrained and seek external financing to cover investment cost I and initial cash reserves C_0. Investment cannot be delayed. Once successfully financed, the firm generates a continuous flow of earnings, with cumulative earnings at time t denoted as X_t. The earnings process is the main state variable and is described in detail in the next subsection. Earnings are subject to corporate taxes at rate τ with a full loss offset provision. The debt coupon payments are deducted from earnings for tax purposes, creating the tax benefit of debt. Corporate cash reserves earn interest at the risk-free rate r. Other interest rates could influence the quantity of cash holdings, but should not affect the economic insights of the model.

The financing may come from a combination of equity and perpetual debt, which promises coupon payments are rate k. The value function of equity is denoted E and that of debt is D. The model allows for both fixed and proportional flotation costs of new issuance, denoted $L \geq 0$ and $\lambda \in [0,1)$, respectively. For the sake of simplicity, the costs are the same for both debt and equity.

The sequence of events and decisions is as follows. At time $t=0$, the firm issues a combination of equity and debt to maximize the value of the original equity holders. After that, the firm starts receiving the flow of earnings and pays out the promised coupon and corporate taxes. Net profits (or losses) are left at the disposal of the firm and are either retained to increase (decrease) cash reserves or are paid out to equity holders as dividends (in the case of instantaneous losses, dividends may be paid out from positive cash reserves). Cumulative dividends up to time t are denoted by Div_t. To deal with indeterminate situations, I assume that equity holders pay out marginal cash holdings whenever they weakly prefer to do so.

When the firm has no means to cover the current coupon payments, it defaults for the reasons of illiquidity. Such an event is called a *liquidity default*. The financial distress is driven here by short-term factors. The firm may also, acting in the interest of equity holders, voluntarily default if the value of equity falls below zero. In this case, the firm is not profitable enough for the equity holders to run it and pay the debt coupons. Then, the firm faces long-term distress; this type of default is referred to as a *solvency default*.

In the event of either type of default, the firm is liquidated, which is costly. The debt claims have the absolute priority in the case of default and the liquidation value is αA, $\alpha \in (0,1)$. Here, $1-\alpha$ is the proportional liquidation cost and A is the value of the all-equity firm at the moment of default.[3]

[3] Following the standard in the literature, we simplify the analysis by assuming that the firm is not refinanced with an optimal capital structure after default.

2.2. Earnings and uncertainty

The firm generates a stochastic flow of earnings before interest and taxes (EBIT):

$$dX_t = \overline{\mu}\, dt + \sigma\, d\overline{Z}_t, \tag{1}$$

where $\overline{\mu}$ is the mean of EBIT, σ is its volatility, and $\overline{Z}$ is a standard Brownian motion. All parties (insiders and outsiders) have the same information at each time t. They observe the cumulative EBIT process $\{X_s, s \le t\}$ that generates a filtration $\{\mathcal{F}_t\}$. There are two sources of uncertainty. First, instantaneous flows are subject to Brownian shocks $d\overline{Z}_t$, which represent short-term liquidity shocks. Second, the profitability of the firm is uncertain, which is represented by the fact that the true mean $\overline{\mu}$ is ex ante unknown to all parties. It is assumed that $\overline{\mu}$ is fixed and can take either of the two values μ_L or μ_H, with $\mu_L < \mu_H$. All parties share a common prior expectation μ_0 about $\overline{\mu}$, with $\mu_0 \in (\mu_L, \mu_H)$.

The two sources of uncertainty serve to capture the two main sides of corporate financial distress. The unpredictable immediate earnings (due to Brownian shocks) bring in the short-term liquidity risk. The uncertain drift $\overline{\mu}$ puts the firm in a position to undergo solvency distress and, ultimately, solvency default.

As time evolves, more information becomes available and the parties update their expectation of mean earnings. The current set of information generated by X_t is described by $\mathcal{F}_t$ and is used in a Bayesian fashion to update the conditional expectation to

$$\mu_t = \mathbb{E}[\overline{\mu} | \mathcal{F}_t].$$

One can use the optimal filtering theory to find the law of motion of the posterior expectation variable. Let an innovation process Z be the difference between the realized and expected earnings; it is defined by the differential equation

$$dX_t = \mu_t\, dt + \sigma\, dZ_t. \tag{2}$$

The process Z is a Brownian motion adapted to filtration $\mathcal{F}_t$. Note that Z differs from $\overline{Z}$ (which is not observable and not adapted to $\mathcal{F}_t$). Eq. (2) describes the dynamics of X in terms of observables.

A version of Theorem 9.1 in Liptser and Shiryaev (2001) then yields that the posterior expectation of the mean earnings level evolves as

$$d\mu_t = \frac{1}{\sigma}(\mu_t - \mu_L)(\mu_H - \mu_t)\, dZ_t. \tag{3}$$

Note first that the posterior expectation process is a martingale as it incorporates all predictable information. Second, the volatility of μ is inversely related to σ, reflecting the fact that expectations adjust more rapidly if the noise term in the earnings process is small (the earnings signals are informative). Finally, learning slows down as evidence accumulates in favor of one state and μ is close to either μ_L or μ_H.

The specification of the cash flow process in (1), which, with the use of filtering theory, can be rewritten as (2) and (3), is the novel and defining feature of the model. The motivation for this modeling choice is threefold.

First, the formulation allows me to capture the key characteristics of corporate liquidity and solvency shocks. Eq. (2) implies that short-term negative (positive) liquidity shocks are more likely if the firm is of low (high) expected long-term profitability. To see this, note that a negative liquidity shock, $dZ_t < 0$, occurs if the time t EBIT dX_t falls below the expected EBIT $\mu_t dt$. This is more probable if the true EBIT rate is low ($\overline{\mu} = \mu_L$) rather than if the rate is high ($\overline{\mu} = \mu_H$). Similarly, a positive liquidity shock, $dZ_t > 0$, is more likely if $\overline{\mu} = \mu_H$. Hence, through the learning mechanism, as Eq. (3) demonstrates, liquidity shocks affect expected profitability. In this way, liquidity and solvency are separate but closely interrelated, or to use a phrase heard from an investment analyst, they are like non-identical twins.

Second, the present paper can integrate two strands of corporate finance literature that have been so far separate. On the one hand, cash flows are subject to unpredictable liquidity shocks to introduce non-trivial cash and dividend policy. This is similar to liquidity management models that analyze optimal dividend policy and predict precautionary cash reserves that cushion liquidity shocks (Jeanblanc-Picqué and Shiryaev, 1995). Technically, cumulative cash flows are modeled here as a stochastic process following an arithmetic Brownian motion. As a result, instantaneous cash flows are *increments* of the process and are subject to Brownian shocks.[4] In contrast, the structural default literature typically models instantaneous cash flows as the *level* of a geometric Brownian motion, in which case instantaneous cash flows are predictable and liquidity management becomes trivial. On the other hand, this model also allows for the drift of the arithmetic Brownian motion to be uncertain to enable endogenous solvency default. In the models based on a simple arithmetic Brownian motion with constant drift, the expected profitability is constant and, given fixed debt obligations, the firm is always either solvent or insolvent. This removes endogenous default from the model. With uncertain drift, as assumed here, the firm may become insolvent, in the sense that it is not profitable enough for equity holders to cover its debt obligations (as in Leland, 1994; Leland and Toft, 1996, and others).

Third, it is analytically convenient to assume cash flows following the stochastic differential Eq. (1). Specifically, I obtain closed-form solutions for corporate securities values, optimal cash reserves, dividends, and a default threshold. The same stochastic environment has been successfully adapted in different contexts by Moscarini (2005) to study job matching in labor markets and Keppo, Moscarini, and Smith (2008) to analyze the value of and demand for information.

[4] Instantaneous cash flows have also been modeled as increments of an arithmetic Brownian motion in the continuous-time agency-based models of corporate finance (DeMarzo and Sannikov, 2006; Biais, Mariotti, Plantin, and Rochet, 2007).

3. Solvency default without liquidity concerns

For the sake of comparison, I start with a benchmark. Following the framework introduced by Leland (1994), assume in this section that the firm is not subject to liquidity default. The endogenous solvency default is triggered by equity holders when equity value becomes negative. The equity holders are willing and able to inject any funds necessary to keep operations running whenever the value of equity is positive. Following Leland (1994), the proceeds from secondary equity financing are not subject to flotation costs. As in numerous contingent claims models of capital structure, a closed-form solution is available under the simplifying assumption that debt is issued only once at the initial date (Leland, 1994; Leland and Toft, 1996; Fan and Sundaresan, 2000; Duffie and Lando, 2001; Miao, 2005; Hackbarth, Hennessy, and Leland, 2007; Sundaresan and Wang, 2007).[5] Accordingly, in this section, assume the following.

Assumption 1. New debt financing is available only at the initial time $t=0$.

Assumption 2. Equity financing is costless beyond $t=0$.

Under these assumptions the firm is without liquidity concerns and there is no room for cash holdings because any liquidity needs can be covered by an injection of equity financing. Subscript *nc* is used in this section with the value functions to denote the financially unconstrained case. For brevity, I suppress the dependence of the value functions on other parameters except for μ, but most notably they also depend on coupon k.

Consider first the value of the firm if it were financed fully by equity. Assuming that $\mu_L \geq 0$,[6] the firm is always profitable and its value is simply equal to the expected discounted future after-tax cash flows:

$$A_{nc}(\mu) = \mathbb{E}_{\mu}\left[\int_0^{\infty} e^{-rt}(1-\tau)\,dX_t\right] = (1-\tau)\frac{\mu}{r}.$$

The liquidation value that debt holders receive in the event of default is $\alpha A_{nc}(\mu)$, with $1-\alpha$ representing the proportional liquidation cost.

The next step is to find the values of the claims held by the debt and equity holders. These values depend on the flows to the claimants and the default time. The optimal default time, chosen by the equity holders, is the first time expected profitability μ falls to some threshold μ^*_{nc}.

The firm issues perpetual debt that pays a constant continuous coupon at rate k per unit of time. It follows from the standard arguments and Ito's lemma that, before default, debt value D_{nc} satisfies the following Bellman-type ordinary differential equation:

$$rD_{nc}(\mu) = \frac{1}{2\sigma^2}(\mu-\mu_L)^2(\mu_H-\mu)^2 D''_{nc}(\mu)+k, \tag{4}$$

subject to

$$D_{nc}(\mu^*_{nc}) = \alpha A_{nc}(\mu^*_{nc}), \quad D_{nc}(\mu_H) = \frac{k}{r}.$$

This system states that if the firm is not in default, the required rate of return on the debt equals the sum of the coupon flow and the expected increase in the value of debt. At μ^*_{nc} the firm defaults and the debt is valued at $\alpha A_{nc}(\mu^*_{nc})$. The boundary condition at μ_H, which is an absorbing state for μ, asserts that D_{nc} is bounded and equal to the risk-free value.

At each period t before default, the equity receives the expected flow of $(1-\tau)(\mu_t-k)$, which is the expected free cash flow after taxes and coupon payments. As in general, $\mu^*_{nc} < k$ (confirmed below in (8)), this means that nonnegative dividends are expected as long as $\mu_t \geq k$ and that in periods with $\mu_t < k$, equity receives "negative dividends" in expectation. The negative distributions are typically interpreted in this type of model as equity issuances. This implies that, unrealistically and inconsistently with evidence on costly equity issuance, the firm resorts to frequent external financing, especially when close to default. This issue is addressed in the main model in Section 4 below.

Within this setting, the equity value E_{nc} must satisfy the following differential equation:

$$rE_{nc}(\mu) = \frac{1}{2\sigma^2}(\mu-\mu_L)^2(\mu_H-\mu)^2 E''_{nc}(\mu)+(1-\tau)(\mu-k), \tag{5}$$

subject to

$$E_{nc}(\mu^*_{nc}) = 0, \quad E_{nc}(\mu_H) = (1-\tau)\frac{\mu_H-k}{r}.$$

This equation and the boundary conditions can be interpreted similarly to the ones for debt valuation.

Having defined equity and debt values, one can calculate total levered firm value F_{nc}, which by definition equals the sum of equity and debt:

$$F_{nc}(\mu) = E_{nc}(\mu)+D_{nc}(\mu). \tag{6}$$

The equity holders choose the default trigger ex post—after the initial financing. This means that they maximize equity value E_{nc} over μ^*_{nc}, which is equivalent to setting the smooth pasting condition on $E_{nc}(\mu)$ at μ^*_{nc}:

$$E'_{nc}(\mu^*_{nc}) = 0. \tag{7}$$

The condition requires the optimal value function to be smooth at the default trigger, and, indeed, it can be shown that it corresponds to the first-order condition from maximization of $E_{nc}(\mu)$ with respect to μ^*_{nc}.

The optimal capital structure is determined at the issuance point with the choice of coupon k, which maximizes the value of the initial equity holders (to indicate the dependence on k directly, it is used explicitly as a parameter of the value functions in the remainder of this section). The firm seeks to finance the investment cost I with debt and new equity. If the new equity holders obtain a fraction $\phi(k)$ of the equity and if the proportional and fixed issuance costs are λ and L, then

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[5] In an alternative and more complex setup, Goldstein, Ju, and Leland (2001) allow for upward leverage adjustments.

[6] The alternative assumption that $\mu_L < 0$ would introduce an optimal liquidation of the firm even in the absence of debt financing. In this case, $A_{nc}(\mu)$ equals the expected discounted future after-tax cash flows up to the time of liquidation, which is optimally chosen by the equity holders. We omit this minor extension, which adds little to our model, while slightly raising the complexity of expressions.

the following financing identity holds:

$$I=(1-\lambda)(D_{nc}(\mu_0,k)+\phi(k)E_{nc}(\mu_0,k))-L,$$

which can be rewritten as

$$(1-\phi(k))E_{nc}(\mu_0,k)=D_{nc}(\mu_0,k)+E_{nc}(\mu_0,k)-\frac{I+L}{1-\lambda}.$$

The left-hand side represents the value of the initial equity holders. Hence, maximization of the left-hand side is equivalent to maximization of $E_{nc}(\mu_0,k)+D_{nc}(\mu_0,k)$. It then follows, using (6), that the optimal choice of coupon k (and thus of the initial leverage) by the initial equity holders is equivalent to maximizing of $F_{nc}(\mu_0,k)$.

The findings of this section are summarized in the following proposition.

Proposition 1. *Suppose Assumptions* 1 and 2 *hold and* $\mu_L \geq 0$. *The optimal solvency default is characterized by the first time* μ *is at or below* μ^*_{nc} *given by*

$$\mu^*_{nc}=\frac{\mu_L\mu_H+[(\beta-1)\mu_H-\beta\mu_L]k}{(1-\beta)\mu_L+\beta\mu_H-k}. \tag{8}$$

If $\mu \geq \mu^*_{nc}$, *the values of equity* $E_{nc}(\mu)$, *debt* $D_{nc}(\mu)$, *and total firm* $F_{nc}(\mu)$ *are given by*

$$E_{nc}(\mu)=(1-\tau)\frac{\mu-k}{r}-\left(\frac{\mu-\mu_L}{\mu^*_{nc}-\mu_L}\right)^{1-\beta}\left(\frac{\mu_H-\mu}{\mu_H-\mu^*_{nc}}\right)^{\beta}(1-\tau)\frac{\mu^*_{nc}-k}{r}, \tag{9}$$

$$D_{nc}(\mu)=\frac{k}{r}-\left(\frac{\mu-\mu_L}{\mu^*_{nc}-\mu_L}\right)^{1-\beta}\left(\frac{\mu_H-\mu}{\mu_H-\mu^*_{nc}}\right)^{\beta}\left(\frac{k}{r}-\alpha A_{nc}(\mu^*_{nc})\right), \tag{10}$$

and

$$F_{nc}(\mu)=(1-\tau)\frac{\mu}{r}+\tau\frac{k}{r}-\left(\frac{\mu-\mu_L}{\mu^*_{nc}-\mu_L}\right)^{1-\beta}\left(\frac{\mu_H-\mu}{\mu_H-\mu^*_{nc}}\right)^{\beta}\left[(1-\alpha)A_{nc}(\mu^*_{nc})+\tau\frac{k}{r}\right], \tag{11}$$

where

$$\beta=\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8r\sigma^2}{(\mu_H-\mu_L)^2}}>1. \tag{12}$$

The optimal coupon rate k^*_{nc} *maximizes* $F_{nc}(\mu_0)$ *over* k.

The closed-form expressions for the value functions are interpreted as follows. The value of equity (9) is the sum of the present value of perpetual distributions to equity and the present value of cash flows lost at default. The value of risky debt in (10) consists of two terms. The first term, k/r, is the value of risk-free perpetual debt. The second term reflects the impact of default risk and equals the present value of cash flows lost by debt in case of default. Total firm value (11) consists of three elements: the first one is the present value of the perpetual flow of net earnings, the second is the present value of the tax benefits of debt, and finally, the negative term corrects for the present value of the cash flows lost at default.

Eq. (8) implies that, in general, $\mu^*_{nc}<k$ (see also the discussion below Proposition 5 and Fig. 3). This means that, as in other structural default models following Leland (1994), the equity holders expect negative cash flows when close to default, yet they prefer to keep the firm running. Moreover, it is worth noting that neither the proportional flotation cost λ nor the fixed one L influences the optimal choice of k.

4. Cash holdings and dividends with financing constraints

As the previous section demonstrates, firms without financing constraints bear no liquidity risk and thus hold no cash reserves. To introduce liquidity risk, the model now restricts the firm's access to external financing. After the initial issuance, which is subject to fixed and proportional costs, the firm cannot raise additional capital. This assumption allows me to find closed-form solutions for the model and obtain a clear-cut comparison between the policies of constrained and unconstrained firms. Within the model, this assumption can be justified by a sufficiently high fixed issuance cost. More generally, financing constraints can be caused by asymmetric information between insiders and outside investors. For example, the firm's insiders may observe cash flows before they are reported to the outsiders. Because the firm knows more about its liquidity and solvency levels, it may be difficult to obtain reasonably priced external financing (similar to Myers and Majluf, 1984). For further reference, the following assumption is introduced.[7]

Assumption 3. New external financing is available only at the initial time $t=0$.

As in the benchmark case, debt holders' claims have absolute priority over the productive assets in the case of default. However, the firm now also holds liquid non-productive assets, namely cash reserves, and it is assumed that these can be distributed to equity before default. The analysis abstracts from any possible contracts that might limit such distributions as its focus is on cash and dividend policies at the discretion of equity holders. In any case, covenants that limit distributions just before default may be difficult to enforce as equity holders would try to preempt them. As shown below, in most cases the optimizing firm reaches the endogenous solvency default trigger with zero cash holdings.

[7] Two more arguments can be given to justify Assumption 3. First, the model focuses on financial distress and constraints are related to external financing of firms in distress. It can well be that external financing of growth opportunities, left unmodeled here, is less constrained. Direct evidence on the significance of financial constraints, especially for firms in distress, is provided by, e.g., Holtz-Eakin, Joulfaian, and Rosen (1994), Zingales (1998), and Campello, Graham, and Harvey (2010).

Second, Assumption 3 replaces Assumptions 1 and 2 of the benchmark model, in which following the standard in the related literature we assumed that, after the initial issuance, equity could be issued frequently and without cost and that the debt flotation costs (or other implicit concerns) would prohibit debt re-issuance. Empirical evidence indicates the opposite: new equity is issued less frequently than debt (Leary and Roberts, 2005) and, if anything, the issuance costs of debt are lower than those of equity (Altinkiliç and Hansen, 2000; Leary and Roberts, 2005). While still simplifying, Assumption 3 may be better in reflecting corporate reality as indicated by the empirical evidence than Assumptions 1 and 2.

4.1. Optimal policies

At each time before default, the firm generates stochastic EBIT dX_t and pays out tax-deductible debt coupon $k\,dt$. The dynamics of earnings net of taxes and debt obligations, denoted by Y_t, is thus

$$dY_t = (1-\tau)(dX_t - k\,dt) = (1-\tau)(\mu_t - k)\,dt + (1-\tau)\sigma\,dZ_t. \quad (13)$$

Without cash reserves and with financing constraints, the firm becomes illiquid and is forced into default as soon as $dX_t < k\,dt$. In this model, positive cash reserves serve as a means to decrease liquidity risk. Denote cash reserves at time t by C_t. Cash reserves change at each time by the instantaneous interest earned on current cash holdings and the difference between net earnings and dividend payout:

$$dC_t = rC_t\,dt + dY_t - dDiv_t. \quad (14)$$

In general, the higher C_t, the lower is the risk of liquidity distress. Of special interest is the level of cash holdings that allows the firm to avoid liquidity default altogether. The next proposition characterizes this level of cash reserves for given coupon k and solvency default trigger μ^* (these values are endogenized in Section 5).

Proposition 2. *Let $\overline{C}$ be the lowest level of cash reserves that allows the firm to avoid liquidity default under Assumption* 3. *$\overline{C}(\mu)$ is given by*

$$\overline{C}(\mu) = (1-\tau)\left[\frac{\sigma^2}{\mu_H - \mu_L}\ln\left(\frac{\mu-\mu_L}{\mu_H-\mu}\frac{\mu_H-\mu^*}{\mu^*-\mu_L}\right) + \max\left\{0, \frac{1}{r}\left(k - \frac{\mu_L+\mu_H}{2}\right)\right\}\right]. \quad (15)$$

The proof, given in the Appendix, relies on the requirement that the dividend process Div_t is non-decreasing. This requirement implies a set of differential equations, with (15) being the minimal solution satisfying these equations.[8]

I show below that cash level $\overline{C}$ plays a key role in optimal liquidity policy. Before interpreting the expression for $\overline{C}$ in (15), it is useful to determine the dividend stream that is implied by the cash policy $C_t = \overline{C}(\mu_t)$. First, by Ito's lemma the dynamics of $\overline{C}$ are

$$d\overline{C}_t = (1-\tau)[\mu_t - \tfrac{1}{2}(\mu_H+\mu_L)]\,dt + (1-\tau)\sigma\,dZ_t. \quad (17)$$

Then using (14) with (17) and (13), the dividend stream is given by

$$dDiv_t = r\overline{C}(\mu_t)\,dt + dY_t - d\overline{C}_t = \left[r\overline{C}(\mu_t) + (1-\tau)\left(\frac{\mu_H+\mu_L}{2} - k\right)\right]dt. \quad (18)$$

As $\overline{C}$ is the lowest level of cash reserves that allows the firm to avoid liquidity default, it is not surprising that $\overline{C}=0$ as μ reaches μ^* in case k is not too large ($k \le \frac{1}{2}(\mu_H+\mu_L)$). If k is larger than $\frac{1}{2}(\mu_H+\mu_L)$, then high coupon payments require positive cash holdings at all times before default. Note that the additional term in (15) when $k > \frac{1}{2}(\mu_H+\mu_L)$, that is $1/r[k - \frac{1}{2}(\mu_H+\mu_L)]$, makes the dividend rate in (18) equal to zero at default.

The explicit formula for $\overline{C}$ in (15) allows for easily calculation of several interesting direct effects of other variables:

$$\frac{\partial\overline{C}}{\partial\mu} > 0, \quad \frac{\partial\overline{C}}{\partial\mu^*} < 0, \quad \frac{\partial\overline{C}}{\partial k} \ge 0, \quad \frac{\partial\overline{C}}{\partial\sigma} > 0.$$

The effects of μ and μ^* are opposite: $\overline{C}$ increases in μ and decreases in μ^*. To see the intuition for this result, note that the difference $\mu - \mu^*$ is a measure of solvency (distance to insolvency). If, with other variables kept constant, μ decreases or μ^* increases, the firm becomes less solvent. Now recall that persistent negative shocks decrease expected profitability and solvency (see Eq. (3)). So a less solvent firm needs to suffer from a less significant series of liquidity shocks before it is considered insolvent. The target level of cash $\overline{C}$ is meant to protect against illiquidity, but not against insolvency. Consequently, lower solvency implies lower $\overline{C}$. The level of debt coupon k affects $\overline{C}$ directly and positively if k is relatively high ($k > \frac{1}{2}(\mu_H+\mu_L)$). This effect is due to the burden that coupon payments impose on cash flows. If k is high, then larger cash holdings are required to complement operational cash flows in meeting high debt obligations. It is important to note, however, that coupon choice will affect $\overline{C}$ also indirectly via the endogenous insolvency trigger μ^*. The analysis will return to the combined effect of k on $\overline{C}$ and its implications in Section 5. The direct effect of EBIT volatility σ on $\overline{C}$ is positive. σ measures the magnitude of liquidity shocks so higher cash reserves are needed to cushion more pronounced shocks. Also in this case, there are other indirect effects; a change in σ will affect the coupon choice and default policy. Section 4.2.3 looks at the total effect and demonstrates, more interestingly, that it remains positive.

Suppose now that the dividend-cash policy aims at decreasing the risk of liquidity default. It is soon verified that this is indeed optimal if the firm's objective is to maximize equity value. Intuitively, this suggests that all cash flows are retained if the firm is at risk of liquidity default and that dividends are paid out as long as such distributions do not bring in liquidity risk. To characterize this proposed dividend policy more formally, denote it by Div_t^* at each time t. If, for a given μ_t, the cash reserves are

[8] An alternative and instructive way to see the result is to think of C_t as the level of cash that is sufficient to withstand a shock in Z_t that brings μ_t to μ^* (irrespective of how quickly the shock is realized). For brevity, we focus here on the case of $k \le \frac{1}{2}(\mu_H+\mu_L)$. Eq. (14) then implies that $\overline{C}(\mu_t) = (1-\tau)\sigma(Z_t - Z^*)$, where $Z_t - Z^*$ is the shock that brings μ_t to default trigger μ^*. To characterize $Z_t - Z^*$, let us define $\theta_t = f(\mu_t) = (\sigma^2/\mu_H - \mu_L)\ln(\mu_t - \mu_L/\mu_H - \mu_t)$ and $\theta^* = f(\mu^*)$ (note that $\theta_t = \theta^*$ if and only if $\mu_t = \mu^*$). Applying Ito's lemma to θ_t, we have

$$\theta_{t'} = \theta_t + \int_t^{t'} \frac{1}{2\sigma^2}(2\mu_s - \mu_H - \mu_L)\,ds + \frac{1}{\sigma}(Z_{t'} - Z_t).$$

This equation also holds for $\theta_{t'} = \theta^*$ in particular. So the shock that brings θ_t to θ^* (and also μ_t to μ^*) is $Z_t - Z^* = \sigma(\theta_t - \theta^*)$. It follows that $\overline{C}(\mu_t)$ must satisfy

$$\overline{C}(\mu_t) = (1-\tau)\sigma(f(\mu_t) - f(\mu^*)) = (1-\tau)\frac{\sigma^2}{\mu_H-\mu_L}\ln\left(\frac{\mu-\mu_L}{\mu_H-\mu}\frac{\mu_H-\mu^*}{\mu^*-\mu_L}\right), \quad (16)$$

which confirms (15) in the proposition for the case $k \le \frac{1}{2}(\mu_H+\mu_L)$. To obtain the additional term in (15), one must impose the condition that the implied dividend payout is not negative for all $\mu_t > \mu^*$ (which is not the case under (16) if $k > \frac{1}{2}(\mu_H+\mu_L)$).

below the target level $\overline{C}$, the firm retains all the earnings:

$$dDiv_t^* = 0 \quad \text{if } C_t < \overline{C}(\mu_t). \tag{19}$$

If the cash level is at $\overline{C}(\mu_t)$, the payout policy is such that this level is maintained as μ_t fluctuates. This is, according to (18)

$$dDiv_t^* = \left[rC_t + (1-\tau)\left(\frac{\mu_H + \mu_L}{2} - k\right)\right] dt \quad \text{if } C_t = \overline{C}(\mu_t). \tag{20}$$

If the cash level exceeds $\overline{C}(\mu_t)$, the residual is paid out

$$dDiv_t^* = C_t - \overline{C}(\mu_t) \quad \text{if } C_t > \overline{C}(\mu_t). \tag{21}$$

Before proving that this cash-dividend policy is optimal for equity holders, it is useful to demonstrate an intuitive property of optimal equity value that states that the partial derivative of the optimal equity value $E(\mu,C)$ with respect to C is larger than or equal to one. This is expected because any extra cash holdings can be paid out immediately as dividends, and the optimal dividend policy can be followed again. To see it, note that for any cash level C, equity value $E(\mu,C)$ of the firm following the optimal dividend policy must be at least equal to the sum of optimal equity value with $C-\Delta C$ cash, $E(\mu,C-\Delta C)$, and ΔC in a dividend payout: $E(\mu,C) \geq E(\mu,C-\Delta C)+\Delta C$. After rearranging the inequality and letting ΔC go to zero, one obtains

$$E_C(\mu,C) \geq 1. \tag{22}$$

The following proposition characterizes the optimal dividend policy.

Proposition 3. *The payout policy* (19)–(21) *maximizes equity value.*

The formal proof of this assertion provided in the Appendix is rather involved, but the basic intuition is fairly straightforward. The proposed payout policy is optimal because it directs the retention of all cash flows whenever marginal cash holdings decrease the probability of illiquidity (so that the cash withheld in the firm is worth more than its face value, $E_C(\mu,C) > 1$) and the payout of excess cash flows otherwise (when marginal cash holdings in the firm are equal to their face value, $E_C(\mu,C) = 1$). A useful corollary of Proposition 3 is that once cash holdings reach $\overline{C}$, then dividend policy is given by (20) and cash balance stays at $\overline{C}(\mu)$ for all levels of μ until solvency default at μ^*. Consequently, the optimal cash and dividends are then described by deterministic functions of μ.

4.2. Implications

This section derives empirical implications with respect to cash and dividend policies and compares the present model to the standard structural models without liquidity concerns.

Changes in exogenous parameters typically affect a number of or all endogenous variables simultaneously. I analyze the comparative statics using the base case as a reference level. The base case parameter values are the following: $\mu_L = 0$, $\mu_H = 0.2$, $\sigma = 0.2$, r=0.06, $\tau = 0.15$, $\alpha = 0.6$, $\lambda = 0.1$, and $\mu_0 = \frac{1}{2}(\mu_H + \mu_L) = 0.1$. The initial value of the expected cash flows is the mean of the binomial distribution. The volatility of cash flows is chosen such that the initial coefficient of variation (that is, σ/μ_0) is equal to 2.0. This corresponds to the annualized coefficients of variation reported in Irvine and Pontiff (2009)—they are equal to 1.59 for cash flows and 2.42 for earnings. The choice of the proportional flotation cost of $\lambda = 0.1$ is above the average parameter value estimated in some other studies (Gomes, 2001; Hennessy and Whited, 2005), and is justified by this paper's focus on firms that are financially constrained beyond the initial issuance. The values of the risk-free rate r, the tax advantage of debt τ, and the recovery rate α closely correspond to the recent calibration exercises for trade-off models; see, for example, Hackbarth, Miao, and Morellec (2006).

4.2.1. *Interaction of liquidity and solvency*

The model identifies several channels that link liquidity and solvency. The first one, referred to as the *information channel*, is a consequence of the EBIT process specified in (1), in which both long- and short-term prospects are uncertain. This gives rise to a filtering problem with the solution in (2) and (3). The dynamics of both the observable EBIT in (2) and profitability in (3) are subject to common shocks. In financial terms, this means that liquidity shocks accumulate to affect solvency levels. If a firm is persistent in liquidity surprises, either positive or negative, they stop being surprising.

The second link works from solvency to liquidity and I designate it as the *hedging channel*. As shown in Section 4.1, changes in solvency (measured by the distance to insolvency $(\mu-\mu^*)$) affect liquidity needs. A less solvent firm has a decreased continuation value and liquidity shocks it is willing to hedge are lower, so the firm needs less cash. In one extreme, for example, a nearly insolvent firm should optimally hold only little cash sufficient to hedge the last negative shocks before insolvency default. The hedging channel also means that for a given level of cash, a decrease in solvency leads to an increase in liquidity.

The information and hedging channels determine the dynamics of cash management and dividend payouts. A third linkage between liquidity and solvency is related to optimal capital structure and will be discussed in Section 5.

It should be noted that the interactions of liquidity and solvency are not specific to firms in financial distress. The dynamics of cash and payouts that these interactions imply are valid for safe firms as well. What is important is that firms are equally financially constrained over time. However, if financing constraints are time-varying (e.g., safe firms become unconstrained), then the strength of the effects may also vary.

4.2.2. *Precautionary cash holdings*

The structural models following Leland (1994) have typically assumed away a meaningful cash policy. As in the benchmark analysis in Section 3, the equity holders are assumed to have no financial constraints and equity issuance is costless. Consequently, any necessary funds are provided by new equity issuance as long as the equity holders are willing to continue operating the firm. This leaves the cash policy irrelevant.

In contrast, the present model predicts that firms hold positive amounts of cash to meet debt coupon payments in case these obligations exceed current earnings. In other words, with costly external financing, cash reserves serve as a cushion to prevent short-term liquidity distress. The key feature of the model is that cash reserves are not meant to cover any losses. If the firm persistently generates losses, the expected profitability decreases and, ultimately, the firm becomes insolvent. As a result, the optimal policy prescribes cash holdings that are a function of the expected earnings and are sufficient to cover liquidity shocks up to the point of endogenous default.

This cash policy has several desirable features and interesting implications, but first it is important to consider whether the role of cash implied by the model reflects the practice of corporate finance. Empirical studies indicate that the demand for corporate cash is driven mainly by the precautionary motives (Opler, Pinkowitz, Stulz, and Williamson, 1999; Lins, Servaes, and Tufano, 2010; Bates, Kahle, and Stulz, 2009). Precautionary cash can serve to fund future growth via capital expenditures and acquisitions (as in, e.g., Almeida, Campello, and Weisbach, 2004; Riddick and Whited, 2009) or to buffer against adverse cash flows. In a study based on financial accounting data, Opler, Pinkowitz, Stulz, and Williamson (1999) find that cash holdings do not seem to be used for capital expenditures, acquisitions, or dividend payments. Instead, large changes in cash are driven by negative or positive cash flow shocks. Based on a recent extensive survey among international chief financial officers (CFOs), Lins, Servaes, and Tufano (2010) conclude about strategic cash (their paper differentiates between operational cash, required in day-to-day operations, and strategic cash, the one studied here and in most of the literature):

> [S]trategic cash serves a basic function—to provide a general purpose buffer against future cash shortfalls. CFOs state that this is the primary driver of strategic cash holdings—with its importance ranking far exceeding that of other response choices. Thus, it appears that firms use strategic cash to insure against all types of negative shocks to cash flows, rather than to just fund growth when external capital may not be available. This finding positions strategic cash holdings as a form of financial distress (or bankruptcy) insurance.

Besides the fact that CFOs do not report funding future investment as an important reason for holding cash, Lins, Servaes, and Tufano (2010) also find that firms that self-report high needs for future external capital hold, in fact. significantly less cash than other firms. Overall, both survey and accounting data evidence closely matches the role of cash that is specified by the model.

It is worth noting that the cash ratio (defined as cash holdings divided by total firm value) implied by the model is in line with cash holdings observed among U.S. firms. With the base case parameters, the cash ratio equals 20.6% and is similar to the average cash ratio of 23.2% shown for a sample of U.S. firms in 2006 by Bates, Kahle, and Stulz (2009).

The model predicts that optimal precautionary cash holdings increase in profitability. This relation is directly explained by the hedging channel linking liquidity and solvency.

A more refined prediction is that cash holdings of financially constrained firms are strongly correlated with cash flows (compare (17) and (2)), while cash holdings of unconstrained firms are not systematically related to cash flows. This implication provides an alternative interpretation of the evidence of Almeida, Campello, and Weisbach (2004) that shows the same pattern of cash flow sensitivity of cash holdings. Almeida, Campello, and Weisbach (2004) explain their findings and precautionary cash holdings by the firms' need to fund future investments while facing financing constraints.[9] In contrast, in the present fully dynamic model, a constrained firm uses positive cash flows to build up cash holdings and uses cash holdings to cover negative cash flows to avoid inefficient default in the future. This mechanism can also be explained by the interaction of liquidity and solvency. Positive cash flow shocks increase the level of solvency via the information channel. Higher solvency results in higher demand for cash via the hedging channel.

4.2.3. Earnings volatility, profitability uncertainty, and cash holdings

This section looks at how the two sources of uncertainty present in the model affect cash holdings. EBIT volatility is related to short-term liquidity risk and profitability uncertainty is related to long-term solvency risk. Fig. 1 examines their effects on the cash ratio.[10]

Increasing EBIT volatility σ has two main direct effects on the endogenous variables. First, it increases the magnitude of liquidity shocks and, thus, liquidity risk. Second, it makes the instantaneous cash flows less informative about the true profitability $\overline{\mu}$. Less informative signals lead to an increase in solvency default trigger μ^* due to a lower value of waiting with the decision to default (see Fig. 3).

Fig. 1A shows that the cash ratio increases in σ. There are a number of forces at work. A larger liquidity risk requires a larger cash buffer. An increase of μ^* in σ means lower solvency and so a lower demand for cash. However, with less informative cash flow signals, the firm must be ready to withstand significant negative liquidity shocks before eventual insolvency, which requires high levels of cash. Fig. 1A indicates that the first and third effects dominate the second one. This prediction is consistent

[9] Khurana, Martin, and Pereira (2006) and Sufi (2009) find further supporting evidence. Riddick and Whited (2009) question these results and, applying a correction in measurement error in Tobin's q, find a negative cash flow sensitivity of cash. Our theoretical contribution can be seen as supporting the positive cash flow sensitivity of cash using a different, arguably more prevalent, motive for corporate cash.

[10] All the results presented in this section hold equally for cash measured in levels.

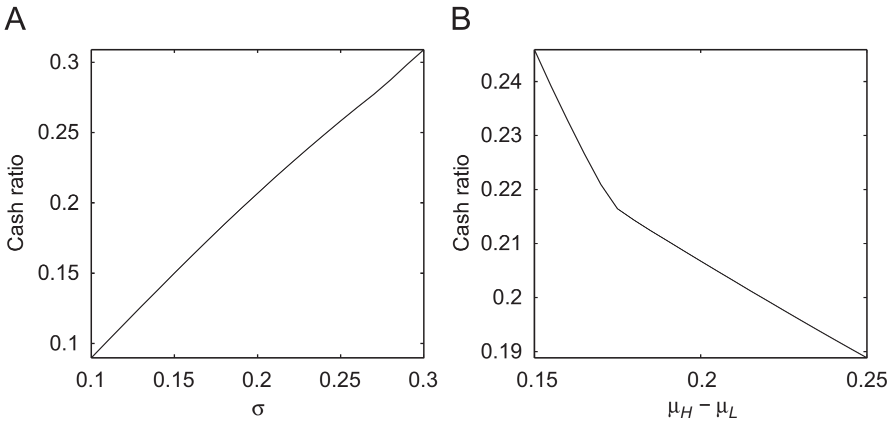

Fig. 1. Effects of EBIT volatility σ and profitability uncertainty $\mu_H - \mu_L$ on the cash ratio. The plotted values are of the financially constrained firm with liquidity concerns and with default and leverage determined endogenously. The other parameter values are $\mu_L = 0$, $\mu_H = 0.2$, r=0.06, $\tau = 0.15$, $\alpha = 0.6$, $\lambda = 0.1$, and $\mu_0 = 0.1$.

with the empirical findings of Opler, Pinkowitz, Stulz, and Williamson (1999) and Han and Qiu (2007). Altogether, the analysis confirms that the explanation in Bates, Kahle, and Stulz (2009), that the recent spectacular expansion in cash holdings among U.S. firms is to a large degree due to the increasing volatility of cash flows, has a theoretical grounding in a model with endogenous cash and financing.

Consider now the effects of changes in the uncertainty about the true level of profitability. With the binomial distribution of $\overline{\mu}$, this uncertainty is measured by the spread between the high (μ_H) and low (μ_L) realizations of mean instantaneous earnings. This variable captures the uncertain economic value of the firm. In the comparative statics exercise, I vary $\mu_H - \mu_L$ around the mean $\mu_0 = \frac{1}{2}(\mu_H + \mu_L) = 0.1$. One effect is that a higher $\mu_H - \mu_L$ increases both the profit and loss potentials of the firm. The other effect is that with a higher spread $\mu_H - \mu_L$, the learning dynamics in μ_t become more rapid as the cash flow signals are more informative about either realization of $\overline{\mu}$ (see Eq. (3)). This leads to a decrease in default trigger μ^* (see Fig. 3).

Fig. 1B shows that cash holdings fall in increasing $\mu_H - \mu_L$. The negative effect comes from the increased speed of learning from cash flow shocks about the expected profitability. If negative liquidity translates quickly in a drop in expected profitability μ_t, then less cash is required to cushion liquidity distress before insolvency at μ^*. It turns out that this effect dominates and $\overline{C}$ falls. This impact of $\mu_H - \mu_L$ on cash levels is opposite to the one of σ and has not been tested empirically.

4.2.4. Smooth dividends

The standard trade-off models treat dividends simply as a balancing item. This leads to a dividend pattern that bears little resemblance to actual corporate payout decisions. As in the benchmark case in Section 3, in these models all positive free cash flows are paid out and dividends are omitted in periods of negative free cash flows.

The model of this paper predicts a very different optimal payout policy. Consistently with empirical observations, firms either do not pay dividends at all (if cash is below $\overline{C}(\mu)$), or, after they initiated distributions, they pay dividends regularly (if cash is at $\overline{C}(\mu)$). When cash reserves are at the target level $\overline{C}(\mu)$, the optimal dividend payout is given by (20). These payouts allow the firm to maintain cash reserves at $\overline{C}(\mu)$ with changing μ. The dividends characterized in (20) are, in contrast to net earnings in (13), without a Brownian shock and, moreover, are strictly positive before default. This implies that dividends are smoothed relative to earnings in line with persistent empirical evidence (Lintner, 1956; Brav, Graham, Harvey and Michaely, 2005; Leary and Michaely, 2008) and that firms in distress would rather reduce dividends but not omit them (DeAngelo and DeAngelo, 1990).

Fig. 2 illustrates the dividend smoothing generated by the model. The left-hand panel presents a simulation of EBIT process X_t and posterior expectations μ_t. I then use the model with liquidity concerns to calculate optimal dividends and cash reserves (the debt coupon and default trigger are set at the optimal levels from the analysis of Section 5). The right-hand panel shows quarterly net earnings and dividends from this simulation. Clearly, the net earnings are positive and negative in different quarters, but these changes are only partly reflected in dividend changes. The dividends remain relatively stable and even in the case of losses, the firm continues to pay out dividends.

The dividend smoothing is driven by the interactions between liquidity and solvency and by the role of cash holdings as a cushion against liquidity shocks. The mechanism can be described as follows. Positive earnings surprises that bring in disposable cash flows also increase expected profitability (the information channel). A more profitable firm is more valuable and thus, it requires more cash reserves to fend off liquidity distress before declaring solvency default (the hedging channel). As a result,

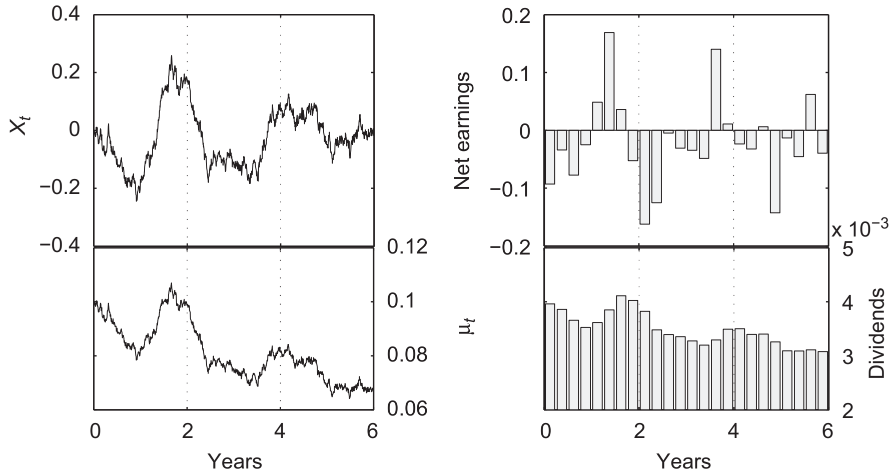

Fig. 2. Simulated quarterly earnings and dividends. Dividends are smoothed compared to earnings. The parameter values are $\mu_L=0$, $\mu_H=0.2$, $\sigma=0.2$, r=0.06, $\tau=0.15$, $\alpha=0.6$, $\lambda=0.1$, and $\mu_0=0.1$.

dividends are flattened in the case of high earnings because an increase in cash flows is offset by increasing optimal cash reserves. In the case of surprisingly low earnings, expected profitability decreases (the information channel), the firm gets closer to endogenous solvency default, and the cash level that allows it to hedge liquidity distress decreases (the hedging channel). Consequently, low earnings lead to a release of some of the cash holdings that are distributed to equity. Both positive and negative earnings surprises are smoothed out, and as Fig. 2 demonstrates, the model predicts positive and stable dividends even if earnings are very volatile.

5. Capital structure, default, and credit spreads

5.1. Valuation of corporate securities

The values of corporate securities depend on a large number of factors, among them the initial cash level financed by external investors. To obtain closed-form solutions, it is assumed that the firm issues securities sufficient to cover cash holdings $\overline{C}(\mu_0)$, which allow the firm to avoid liquidity risk.

Assumption 4. $C_0=\overline{C}(\mu_0)$.

Note that this assumption is partially validated by Assumption 3, which constrains the availability of external financing to the initial date. Without additional external financing, all the required cash is raised with the initial issuance.[11] If $C_0=\overline{C}(\mu_0)$, then by Proposition 3 the optimal dividend policy is given in (20) for all $\mu_t>\mu^*$. This payout policy implies that $C_t=\overline{C}(\mu_t)$ for all $\mu_t>\mu^*$. In other words, under Assumption 4, the firm holds cash reserves at the level $\overline{C}(\mu_t)$ until the endogenous solvency default and is hedged against liquidity risk.

Under the assumptions of the model, debt value D equals the present value of continuous coupon payments up to the time of default as soon as μ_t reaches μ^*. $D(\mu)$ must satisfy the following differential equation:

$$rD(\mu)=\frac{1}{2\sigma^2}(\mu-\mu_L)^2(\mu_H-\mu)^2D''(\mu)+k.$$

At default, debt holders receive a fraction α of the EBIT-generating technology. That is, following the earlier literature, the model simplifies the financing issues after default. This implies that the debt holders recover $\alpha A(\mu^*)$ at default, where $A(\mu)=\alpha(1-\tau)\mu/r$ if $\mu_L\geq 0$. Thus, the differential equation for D is coupled with the following boundary conditions:

$$D(\mu^*)=\alpha A(\mu^*),\quad D(\mu_H)=\frac{k}{r}.$$

Before default at the first time μ_t falls to μ^*, the equity receives a flow of dividends that is equal to (combining (20) and (15))

$$dDiv_t=a_1\ln\left(\frac{\mu-\mu_L}{\mu_H-\mu}\frac{\mu_H-\mu^*}{\mu^*-\mu_L}\right)dt+a_2\,dt,$$

where

$$a_1=\frac{(1-\tau)r\sigma^2}{\mu_H-\mu_L},$$

and

$$a_2=(1-\tau)\max\left\{0,\frac{\mu_L+\mu_H}{2}-k\right\}.$$

Then it follows from the standard arguments that equity value E must satisfy the ordinary differential equation:

$$rE(\mu)=\frac{1}{2\sigma^2}(\mu-\mu_L)^2(\mu_H-\mu)^2E''(\mu)+a_1\ln\left(\frac{\mu-\mu_L}{\mu_H-\mu}\frac{\mu_H-\mu^*}{\mu^*-\mu_L}\right)+a_2, \tag{23}$$

[11] Note that with a sufficiently high variable issuance cost λ, the firm might prefer issuing securities for less than $I+\overline{C}(\mu_0)$ (but more than I) and collecting the remaining cash up to $\overline{C}$ from the retained earnings. The firm would balance the cost of exposure to liquidity risk and the benefit of a cheaper source of capital. We focus on the case generated by Assumption 4 that results in closed-form solutions for equity and debt values.

subject to the following boundary conditions:

$$E(\mu^*)-\overline{C}(\mu^*)=0, \quad E(\mu_H)-\overline{C}(\mu_H)=(1-\tau)\frac{\mu_H-k}{r}. \tag{24}$$

As usual, the left-hand side of (23) reflects the required rate of return per unit of time for holding equity. The right-hand side represents the expected change in equity value plus the dividend flow per unit of time. The boundary condition at μ^* states that the value of equity net of cash is zero at insolvency and is in line with the assumption that the equity holders can withdraw non-productive liquid assets prior to default. The boundary condition at μ_H ensures that $E(\mu_H)-\overline{C}(\mu_H)$ is bounded and equal to the risk-free value of free cash flows.

Solving the respective differential equations with the boundary conditions, one can obtain closed-form solutions for both equity and debt values. The following proposition shows these results.

Proposition 4. *Suppose Assumptions* 3 and 4 *hold. Then, for a given default trigger μ^* and $\mu \geq \mu^*$, debt and equity value satisfy*

$$D(\mu)=\frac{k}{r}-\left(\frac{\mu-\mu_L}{\mu^*-\mu_L}\right)^{1-\beta}\left(\frac{\mu_H-\mu}{\mu_H-\mu^*}\right)^{\beta}\left(\frac{k}{r}-\alpha A(\mu^*)\right), \tag{25}$$

and

$$E(\mu)=\overline{C}(\mu)+(1-\tau)\frac{\mu-k}{r}-\left(\frac{\mu-\mu_L}{\mu^*-\mu_L}\right)^{1-\beta}\left(\frac{\mu_H-\mu}{\mu_H-\mu^*}\right)^{\beta}(1-\tau)\frac{\mu^*-k}{r}, \tag{26}$$

with β given in (12).

Eq. (25) implies that, for a given coupon k and default trigger μ^*, the value of debt is the same as in the benchmark case reported in Eq. (10) and is equal to the sum of the value of risk-free debt and the present value of the loss at default. Eq. (26) reveals that the value of equity, which is the present value of the flow of dividends until default, can be decomposed into three elements. It is a sum of the value of corporate cash plus the present value of all future net earnings plus the value of the option to default on debt of the insolvent firm at μ^*. Notably, despite the fact that optimal dividends are different than net earnings, the equity value consists of the discounted value of net earnings plus current cash holdings. The reason for this is that even with liquidity concerns, the equity holders remain, in expectation, the claimant of all net earnings before default, but they use cash as a buffer between net profits and dividends to time distributions appropriately to manage liquidity risk.

The total firm value F equals the sum of the value of equity and the value of corporate debt. From Proposition 4, it follows that, if $\mu_L \geq 0$,

$$F(\mu)=E(\mu)+D(\mu)=\overline{C}(\mu)+(1-\tau)\frac{\mu}{r}+\tau\frac{k}{r} -\left(\frac{\mu-\mu_L}{\mu^*-\mu_L}\right)^{1-\beta}\left(\frac{\mu_H-\mu}{\mu_H-\mu^*}\right)^{\beta}\left[(1-\alpha)(1-\tau)\frac{\mu^*}{r}+\tau\frac{k}{r}\right]. \tag{27}$$

Eq. (27) demonstrates that the firm value is a sum of four components. It consists of the face value of cash holdings plus the present value of earnings net of taxes plus the present value of tax shield of debt minus the probability-adjusted present value of cash flows lost at default. Using (6) in (27) shows that the firm value net of cash, $F(\mu)-\overline{C}(\mu)$, is equal to $F_{nc}(\mu)$, that is, the firm value of the firm with no financing constraints. By holding cash reserves $\overline{C}(\mu)$, the firm is hedged against liquidity distress and thus, the value of its productive assets is equal to those of the financially unconstrained firm. Moreover, the cash in the firm $\overline{C}(\mu)$ is worth exactly $\overline{C}(\mu)$ because the interest gained on cash equals the investors' discount rate. However, this close relation between the values of constrained and unconstrained firms holds only for given common debt levels (if k's are equal). But as the next section shows, the constrained firm with liquidity concerns chooses different financial leverage than the firm with no financing constraints.

5.2. Default and optimal capital structure

Under Assumptions 3 and 4, the firm uses cash reserves to hedge against liquidity shocks. Then the timing of default is endogenously selected by the equity holders. Default takes place at the moment that the firm is not solvent enough. The default policy takes the form of a lower threshold on μ, which maximizes equity value. This is achieved at μ^*, which satisfies the smooth pasting condition:

$$E'(\mu^*)=\overline{C}'(\mu^*). \tag{28}$$

(Compare it with the smooth pasting condition (7) and the boundary condition for E at $\mu=\mu^*$ in (24) in the present model.)

The initial equity holders use equity and debt to finance the investment cost I and the initial level of cash reserves $\overline{C}(\mu_0,k)$ (to stress the dependence on k, I add parameter k to cash and value functions in the rest of this section). If the new equity holders obtain a fraction $\phi(k)$ of equity and if the proportional cost of issuance of both debt and equity is λ and the fixed cost of issuance is L, then the following financing identity holds:

$$I+\overline{C}(\mu_0,k)=(1-\lambda)(D(\mu_0,k)+\phi(k)E(\mu_0,k))-L.$$

This can be rewritten as

$$(1-\phi(k))E(\mu_0,k)=D(\mu_0,k)+E(\mu_0,k)-\frac{\overline{C}(\mu_0,k)}{1-\lambda}-\frac{I+L}{1-\lambda}.$$

The left-hand side represents the value to the initial equity holders. It follows that the optimal k that maximizes $(1-\phi(k))E(\mu_0,k)$, also maximizes the right-hand side, and the objective function can be expressed as (30) in the next proposition. The same proposition also presents the solution to the smooth pasting condition (28) for the optimal default trigger.

Proposition 5. *Under Assumptions* 3 and 4, *the optimal solvency default is characterized by the first time that μ is at or below μ^*, given by*

$$\mu^*=\frac{\mu_L\mu_H+[(\beta-1)\mu_H-\beta\mu_L]k}{(1-\beta)\mu_L+\beta\mu_H-k}. \tag{29}$$

The optimal debt coupon rate k^* maximizes

$$F(\mu_0,k)-\frac{\overline{C}(\mu_0,k)}{1-\lambda} \tag{30}$$

over k.

Fig. 3 presents the main properties of the optimal default trigger function (29). μ^* is a convex increasing function of k. It is intuitive that μ^* is equal to μ_H (μ_L) with coupon equal to μ_H (μ_L). This is because, with $k=\mu_H$, the equity holders expect losses for all μ and thus, default immediately with $\mu^*=\mu_H$. When $k=\mu_L$, the firm generates positive expected profit net of coupon for all μ except at the absorbing state at μ_L, and thus, the equity value is maximized with a default at $\mu^*=\mu_L$. For the intermediate values of k in (μ_L,μ_H), the default threshold falls below the coupon rate; in the figure, μ^* lies below the diagonal $\mu^*=k$. This difference between the expected earnings at default and coupon represents the value of waiting to default. Because of this value, the equity holders prefer to keep the firm running despite that the coupon obligations exceed the expected earnings.

As illustrated in Fig. 3, default triggers μ^* increase in β. By Eq. (12), β depends on the earnings signal quality (that is, on σ and on $\mu_H-\mu_L$) and the discount rate. It follows that the default trigger increases with the noisiness of the earnings signals (higher σ or smaller $\mu_H-\mu_L$) and with the level of discount rate r. Intuitively, with noisy signals and high r, the value of postponing default in order to wait for new information decreases.

μ^* in Eq. (29) is the same as μ^*_{nc} in the benchmark case reported in (8). Since the firm is effectively hedged against liquidity distress, it makes sense that the solvency default trigger that maximizes equity value is the same as for the financially unconstrained firm. Interestingly, this is despite the precautionary cash reserves that need to be held in the firm. However, the isomorphism of μ^* and μ^*_{nc} means only that the default policy in both cases is the same if coupon obligations are the same. The second part of Proposition 5 implies that, in general, the optimal coupons differ in the two cases with and without liquidity concerns.

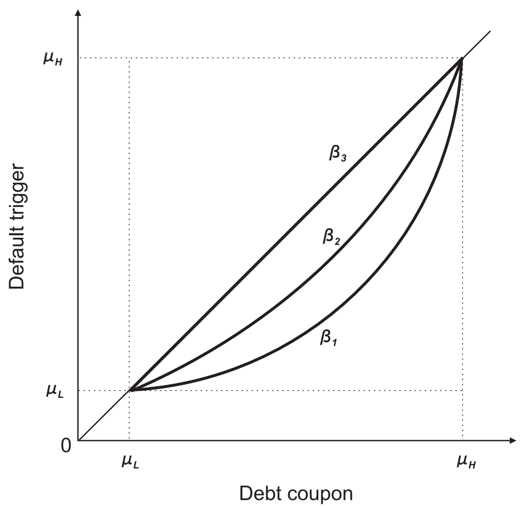

Fig. 3. Optimal default trigger μ^* as a function of debt coupon k for various values of β with $\beta_1<\beta_2<\beta_3$ and $\beta_3\to\infty$. β increases in σ and r and decreases in $(\mu_H-\mu_L)$.

Using (27), the objective function (30) can be rewritten as

$$F_{nc}(\mu_0,k)-\frac{\lambda}{1-\lambda}\overline{C}(\mu_0,k). \tag{31}$$

Comparing this objective function with the one of the financially unconstrained firm (which was $F_{nc}(\mu_0,k)$), one notes the major difference between the cases. Whereas the coupon choice in the benchmark analysis was independent of any issuance cost, the optimal coupon of the constrained firm depends on the proportional issuance cost λ. This is because now the capital structure choice interferes with the firm's financing needs: the firm needs to raise capital to cover the initial cash balance, and the required cash balance depends on the coupon rate itself. As raising additional units of cash is costly due to the variable issuance cost, the firm's optimal choice of debt also takes into account its impact on the initial amount of cash to be raised. One can expect that the outcome depends on the sign of the relation between k and $\overline{C}$. If $\overline{C}$ decreases in k, the firm should be willing to accept a higher coupon to limit the needed cash and save on the cost of raising additional capital. If, on the other hand, $\overline{C}$ increases in k, it should be optimal to take somewhat less coupon and debt. I shall analyze these effects and their consequences in Section 5.3.

In the model, liquidity risk can be hedged with appropriate cash policy. However, the presence of liquidity concerns is sufficient to distort the financing policies and the firm value. The value of the constrained firm net of cash ($F-\overline{C}$) is always less than (or equal to, is some special cases) the value of the unconstrained firm (F_{nc}):

$$F(\mu_0,k^*)-\overline{C}(\mu_0,k^*)\le F_{nc}(\mu_0,k^*_{nc}).$$

The relevant comparison is with the constrained firm net of cash because the unconstrained firm does not hold cash in the model. The reason for the inequality is that the unconstrained firm chooses its debt coupon $k_{nc}{}^*$ to maximize $F_{nc}(\mu,k)$. By (27), the same coupon maximizes $F(\mu,k)-\overline{C}(\mu,k)$. However, the constrained firm selects its coupon k^* to maximize (30), $F(\mu,k)-\overline{C}(\mu,k)/(1-\lambda)$. It follows that the capital structure of the constrained firm is distorted in such a way that its net value is below that of the unconstrained firm.

It is worth noting that, in the absence of financing frictions in the sense of zero variable cost of issuance ($\lambda=0$), the objective function simplifies to $F_{nc}(\mu_0,k)$ and is exactly equivalent to the problem in the case without liquidity constraints. Moreover, the fixed cost of issuance does not matter for the choice of the optimal k.

5.3. Implications

To examine implications of the model with respect to capital structure and credit spreads, I employ the same base case parameters as in Section 4.2.

5.3.1. Interaction of liquidity and solvency

Section 4.2 introduced the information and hedging channels linking liquidity and solvency. Endogenous leverage generates a third linkage; liquidity concerns affect the level of solvency via the capital structure choice. Accordingly, this effect is referred to as the *leverage channel*. It originates from the interaction between capital structure and the demand for external financing. As discussed above, the constrained firm raises capital to cover not only the required investment outlay but also the initial cash reserves, and the level of cash is affected by the amount of issued debt. Because marginal external financing is costly, the firm, in its capital structure choice, will attempt to minimize the initial level of cash (see Eq. (30)).

To understand the direction of this mechanism, consider the impact of debt coupon rate k on cash holdings. From (15) observe that k affects $\overline{C}$ in two ways. The main effect works for all levels of k indirectly via solvency default trigger μ^*. Higher coupon obligations mean closer insolvency (that is, higher μ^*; see (29)) and this results in lower cash needs. The direct effect, discussed already in Section 4.1, comes from the last term of (15) and stems from the fact that high debt obligations deplete cash flows. It works if debt coupon is relatively high ($k > \frac{1}{2}(\mu_H + \mu_L)$) and results in $\overline{C}$ increasing in k. Fig. 4 demonstrates the effects of coupon on the target level of cash $\overline{C}$ for the base case environment. The total impact is such that cash holdings decrease in k for small k and increase in k for larger k, and it appears robust for various parameter choices. Because $\overline{C}$ is minimized at the intermediate levels of k, it follows that to minimize the flotation cost of raising the initial cash reserves, the constrained firm issues more debt than the unconstrained firm if the unconstrained firm's optimal coupon is relatively low (below $(\mu_L + \mu_H)/2$). The opposite happens if the unconstrained firm's optimal coupon is high (above $(\mu_L + \mu_H)/2$).

The leverage channel can be viewed as an extension of the trade-off theory. In the standard trade-off theory without liquidity concerns, capital structure is determined to balance tax benefits of debt and bankruptcy costs. With liquidity concerns, firms must take into account another trade-off layer. Firms do not want to take too little debt because high solvency exposes them to high liquidity risk. This requires firms to raise more initial cash holdings, which is costly. On the other hand, firms do not want to accept too high debt levels as this implies high coupon payments that put a strain on cash flows. In this case, firms need more cash holdings to complement cash flows in case of liquidity shocks—and this is again costly.

5.3.2. Cash holdings and debt

The empirical literature has been interested in the impact of debt on corporate cash holdings, treating the former variable as exogenous. Fig. 4 presents the cash level $\overline{C}$ as a function of debt coupon and shows that cash decreases in debt for low and moderate levels of debt and increases with high levels of debt. The empirical evidence of Opler, Pinkowitz, Stulz, and Williamson (1999) shows a negative relation between cash and leverage. A more refined study by Guney, Ozkan, and Ozkan (2007) provides evidence for a non-monotonic relation between cash holdings and debt, in line with this paper's prediction.

The model predicts that the marginal value of cash holdings to equity holders varies across firms with different capital structures. In particular, the model is able to encompass all the main hypotheses of the recent empirical study of Faulkender and Wang (2006). They hypothesize and empirically show that the marginal value of cash is higher for financially constrained firms and is decreasing in the level of cash reserves and the leverage ratio. In the present framework, the marginal value of cash is equal to one for both unconstrained and constrained firms at or above the target cash level $\overline{C}$. Because the probability of liquidity default decreases with an additional unit of cash, the marginal value of cash exceeds one in constrained firms with cash below $\overline{C}$.

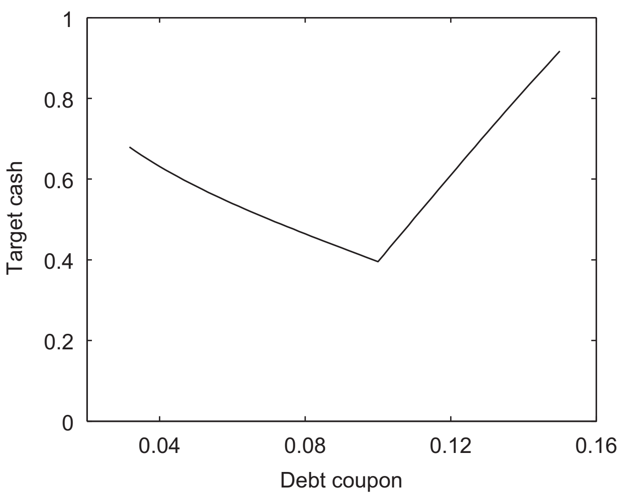

Fig. 4. Cash balance $\overline{C}$ as a function of debt coupon k. The parameter values are: $\mu_L = 0$, $\mu_H = 0.2$, $\sigma = 0.2$, r=0.06, $\tau = 0.15$, $\alpha = 0.6$, and $\mu_0 = 0.1$.

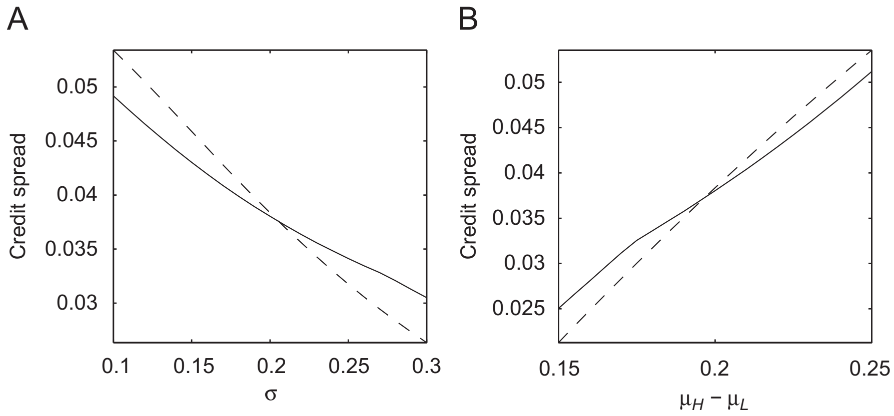

Fig. 5. Effects of EBIT volatility σ and profitability uncertainty $\mu_H-\mu_L$ on credit spreads. The solid lines plot the respective values of the financially constrained firm with liquidity concerns, and the dashed lines plot the values of the unconstrained firm. Default and leverage are determined endogenously. The other parameter values are $\mu_L=0$, $\mu_H=0.2$, r=0.06, $\tau=0.15$, $\alpha=0.6$, $\lambda=0.1$, and $\mu_0=0.1$.

It follows that the marginal value of cash is larger for constrained firms and that, in the case of constrained firms, it decreases with the level of cash holdings. Most interestingly, one can derive a clear interpretation of the negative cross-sectional relation between the marginal value of cash and debt level found by Faulkender and Wang (2006) (they seem to build their hypothesis and interpretation on the contingent claims models that do not have a meaningful cash policy). As explained above, for small and moderate levels of debt, the target level of cash decreases in debt. Then, for a fixed level of cash below $\overline{C}$, an increase in debt implies that the current cash holdings are closer to $\overline{C}$ so the firm is closer to being fully hedged against liquidity risk. Consequently, the marginal value of cash decreases in debt. The model also predicts an untested possibility that the relation is reversed for high levels of debt.

5.3.3. *Earnings volatility, profitability uncertainty, and credit spreads*

This section analyzes the effects of the two sources of uncertainty on credit spreads. As in Section 4.2.3, EBIT volatility is measured by σ. Profitability uncertainty is varied by a mean preserving spread of $\mu_H-\mu_L$ around 0.1. Fig. 5 displays their effects on credit spreads defined by the difference between the debt yield and the risk-free rate, $k/D-r$. The presented values are calculated with default triggers and coupons at the optimal levels.

The total effect of EBIT volatility σ on credit spreads is negative (see Fig. 5A). Higher volatility magnifies liquidity shocks and makes cash flow signals less informative about profitability. Higher liquidity risk and lower informativeness of cash flows increase the cost of debt. The opposing effect decreasing credit spreads is that the firm responds to the more expensive debt financing by issuing less debt. It turns out that the second effect dominates.

Changes in the uncertainty about the true level of profitability affect credit spread in several ways. A higher spread $\mu_H-\mu_L$ means that cash flow signals are more informative and that default is relatively late. These effects make debt cheaper, but they may be offset if more debt is issued. As demonstrated in Fig. 5B this is the case, credit spreads increase in $\mu_H-\mu_L$ because the combined effects of the decreased default trigger and of informative cash flows make it attractive for shareholders to issue more debt.

It is interesting to investigate how cash and credit spreads are related to each other when the exogenous variables vary (that is, combine Fig. 1A with Fig. 5A and Fig. 1B with Fig. 5B). It appears that credit spreads decrease in cash. This pattern is persistent and irrespective of whether the underlying exogenous variable is σ or $\mu_H-\mu_L$. Empirically, such a negative relation is found in Acharya, Davydenko, and Strebulaev (2008).

When comparing the effects of earnings volatility and profitability uncertainty in Figs. 1 and 5, it is striking that the two measures of risk have the opposing signs for some of the key financial variables.[12] In essence, this is because EBIT volatility and profitability uncertainty are differently related to liquidity and solvency concerns. Taken together, the results presented here call for a differentiation between short-term volatility in cash flows and long-term uncertainty about economic prospects in both theoretical and empirical analysis of corporate finance.

5.3.4. *Dispersion of credit spreads*

The model has further implication for debt credit spreads when it is compared to the benchmark model without liquidity concerns. In Fig. 5, the solid lines plot the values for the financially constrained firm with liquidity concerns, and the dashed lines plot the values for the unconstrained firm. Both Figs. 5A and B show that with financing constraints, the predicted credit spreads

[12] The effects are not always opposite for other variables. In unreported analysis, we find, for example, that the leverage ratio decreases in both measures of uncertainty. This relation is in accordance with the empirical evidence on leverage (Titman and Wessels, 1988).

are less dispersed than in the case without financing constraints.

This result is explained by the leverage channel that links liquidity and solvency, that is, by the influence of exposure to liquidity risk on optimal leverage. As discussed before, because external financing is costly, firms attempt to not raise too much initial liquidity and this is achieved at intermediate debt levels. Hence, high debt levels are decreased and low debt levels are increased, which ultimately translates into flattened credit spreads when compared to the financially unconstrained case.

This effect allows the model to address the key problem with the predictive power of structural models as reported by Eom, Helwege, and Huang (2004). They test the yield spread predictions of several structural models and conclude that the available models tend to produce too high a dispersion of predicted credit spreads. Where the structural models predict high credit spreads, these predictions notably exceed the actual spreads, and where the models predict low credit spreads, these predictions fall significantly below the observed ones. A closely related regularity is shown by Huang and Huang (2003), who find that a small (large) fraction of yield spreads of investment-grade bonds (high-yield bonds) is explained by credit risk implied by structural models. Liquidity concerns in this model shift the predicted credit spreads in the desired direction.

5.3.5. Leverage

A weakness of the standard trade-off model of capital structure that has frequently been raised in the literature is that the optimal leverage implied by the model exceeds the leverage ratios observed empirically. The model proposed here lessens this problem. Numerical analysis indicates that the leverage ratio (debt to firm value) of the firm with liquidity concerns is significantly below the ratio of the unconstrained firm. For different parameter values, the drop in the leverage ratio is between 15% and 40%. For example, with the base case parameters, the unconstrained firm's leverage ratio is 0.68 and falls to 0.53 for the firm with liquidity concerns. While there are a number of effects that liquidity concerns bring to capital structure, the driving force behind this remarkably reduced leverage is the recognition of the role of cash in corporate assets. As the total assets of the constrained firm incorporate the value of cash, the leverage ratio decreases.

6. Related literature

This paper builds on the contingent claims models of risky asset valuation introduced by Black and Scholes (1973) and Merton (1974). Since the trade-off models of Fischer, Heinkel, and Zechner (1989) and Leland (1994), an important part of the literature has focused on the corporate-finance implications of contingent claims modeling with the central role given to the optimal choice of capital structure. Subsequent extensions analyze debt maturity, debt renegotiation, recapitalization, incomplete accounting information, macroeconomic regimes, debt structure, and investment.[13] Despite these developments, the structural trade-off framework has not been successful in incorporating some essential corporate financial decisions. The existing models typically predict no role for corporate cash holdings, treat dividends merely as balancing items, and focus on solvency default and neglect liquidity concerns. The contribution of this paper is to provide a tractable model of dynamic cash and dividend policies with realistic treatment of liquidity and solvency concerns.

An exception within the structural trade-off literature is the paper of Anderson and Carverhill (2007). Like this model, theirs also features two sources of uncertainty in cash flows. However, because the uncertainties are left independent, their analysis does not share with this paper the richness of interactions between liquidity and solvency and the predictions with respect to cash, dividends, and credit spreads. Instead, they employ numerical techniques and focus on dynamic refinancing.

This paper is also related to the literature on dynamic liquidity management and dividend payout optimization. Jeanblanc-Picqué and Shiryaev (1995) study a tractable model of a financially constrained firm threatened by costly liquidation, in which the optimal payout policy is to retain all earnings if cash reserves are below a certain fixed threshold and to pay out everything otherwise. The model has been extended to incorporate, among others, risk management, investment, and costly financing (Højgaard and Taksar, 1999; Décamps and Villeneuve, 2007; Décamps, Mariotti, Rochet, and Villeneuve, 2008). This paper shows that adding uncertainty in the expected level of cash flows and concerns over solvency leads the optimizing firm to smooth dividends relative to cash flows.

The analysis here is also related to DeMarzo and Sannikov (2008). In their model, an agent controls the firm's expected cash flows through costly effort, and the initially unknown expected profitability is learned over time. While their approach is different than mine, the optimal compensation contract in their model specifies payments that are smoothed over cash flows as are equity-value-maximizing dividends in this paper.

Several recent papers also feature both cash holdings and debt financing. Hennessy and Whited (2005) present a trade-off model in which firms use a mix of equity, one-period debt, and cash balance to cover their financing needs. In contrast to this model, in Hennessy and Whited (2005) default is precluded, which results in riskless debt and zero credit spreads, and firms never hold both debt and positive cash balance at the same time. Moreover, the analysis here is focused on the roles of short-term liquidity and long-term solvency distresses, while the framework of Hennessy and Whited (2005) does not model and distinguish these forces. Gamba and Triantis

[13] See Leland and Toft (1996), Fan and Sundaresan (2000), Goldstein, Ju, and Leland (2001), Duffie and Lando (2001), Hackbarth, Miao, and Morellec (2006), Broadie, Chernov, and Sundaresan (2007), Hackbarth, Hennessy, and Leland (2007), and Sundaresan and Wang (2007), among others.

(2008) extend Hennessy and Whited (2005) and allow firms to hold both debt and cash holdings at the same time, but the other differences remain. Acharya, Almeida, and Campello (2007) recognize that, as in this paper, the presence of financing frictions is a precondition for a meaningful role of cash holdings in corporate policy. Their motivation for cash is, however, based on the distinct roles of cash and negative debt in hedging future investment opportunities against future cash flows. Acharya, Huang, Subrahmanyam, and Sundaram (2006) introduce cash holdings into a discrete-time model of risky debt. Their focus is on the role of strategic debt renegotiation. Overall, the analysis in this paper with closed-form results is more tractable than previous models that relied on numerical solutions.

7. Conclusions

Earlier literature has studied either solvency distress with optimal capital structure or liquidity distress with cash and dividend policy. The analytically tractable framework presented in this paper allows one to study both sources of financial distress simultaneously and to explore the interplay of financing, cash, and dividends.

I find that corporate liquidity and solvency interact through information, hedging, and leverage channels. These interactions can help to explain several empirical regularities. The information and hedging channels cause equity-maximizing firms to smooth dividends and to absorb cash flow shocks in cash holdings. The leverage channel, which captures the fact that firms select their leverage to limit exposure to liquidity risk, can explain a low dispersion of credit spreads found in empirical studies. I further find that long-term profitability uncertainty, measuring solvency risk, and cash flow volatility, measuring liquidity risk, can have opposing effects on various variables. These findings suggest that empirical studies should pay attention to the effects of uncertainty besides the usual focus on volatility (see, e.g., Anderson, Ghysels, and Juergens (2009) for an empirical proxy of long-term uncertainty).

The model can be extended to study a number of additional issues, which are left for future research. First, it would be interesting to analyze dynamic capital structure choice with different degrees of financing constraints. If debt and equity refinancing is costly, then the decision whether to finance liquidity needs or to default due to illiquidity might depend on the level of solvency. Second, the paper considers a single firm with exogenously determined cash flows. Competition can affect firms' cash flows and default strategies, and thus optimal leverage, demand for cash, and also dividends. Analyses of both oligopoly and competitive equilibria may be worth pursuing (Lambrecht, 2001; Miao, 2005). Third, future work can also analyze the role of changing macroeconomic conditions in the framework of this paper. Macroeconomic risk has been recently successfully incorporated into contingent claims models of capital structure (Hackbarth, Miao, and Morellec, 2006; Bhamra, Kuehn, and Strebulaev, 2010; Chen, 2010) and may also have an important impact on corporate liquidity and solvency risks and their interaction.

Finally, a promising direction for future research would be to introduce asymmetric information between corporate insiders and outside investors. It is likely that outsiders observe true cash flows but with a lag. It could be also that the firm's insiders know the true profitability before (alternatively, after) they seek external financing whereas investors cannot observe it directly. Each of these situations might create adverse selection problems that could deepen financial constraints.

Appendix A. Proofs

Proof of Proposition 1. I first solve for the equity value function. Differential equation (5) has an analytical solution of the following general form:

$$E_{nc}(\mu)=B_1(\mu-\mu_L)^{1-\beta}(\mu_H-\mu)^{\beta}+B_2(\mu-\mu_L)^{\beta}(\mu_H-\mu)^{1-\beta}+(1-\tau)\frac{\mu-k}{r}, \tag{32}$$

where $\beta>1$ is the positive root of

$$\beta^2-\beta-\frac{2r\sigma^2}{(\mu_H-\mu_L)^2}=0,$$

and B_1, B_2 are constants that are determined by boundary conditions. The first two terms constitute the general solution to the homogeneous part of (5) and the third term is an easy-to-guess particular solution to the whole non-homogeneous equation (5). The boundary condition at μ_H implies that B_2=0. This is because, with $\beta>1$ for any other B_2, $E_{nc}(\mu_H)$ is unbounded. Using the boundary condition at μ^*_{nc} to determine B_1 delivers the expression for $E_{nc}(\mu)$ given in the proposition.

Debt value is found analogously using that the general solution to differential equation (4) is

$$D_{nc}(\mu)=B_3(\mu-\mu_L)^{1-\beta}(\mu_H-\mu)^{\beta}+B_4(\mu-\mu_L)^{\beta}(\mu_H-\mu)^{1-\beta}+\frac{k}{r},$$

with β as above and constants B_3 and B_4. Applying the boundary conditions on D_{nc} at μ_H and μ^*_{nc} yields (10). Firm value F_{nc} given in (11) follows by adding (9) and (10).

Optimal default trigger μ^*_{nc} in (8) is delivered by applying the smooth pasting condition (7) to Eq. (9). □

Proof of Proposition 2. For an arbitrary function $\overline{C}(\cdot,\cdot)$, let $C_t=\overline{C}(\mu_t,X_t)$, so that C_t is allowed to depend on both state variables. Denote the default time associated with trigger μ^* by $t^*=\inf\{t\geq 0:\mu_t<\mu^*\}$. The firm is liquid up to time t^* if $C_t\geq 0$ for all $t\leq t^*$. Note that, for example, a simple cash policy C_t=0, $t\leq t^*$, satisfies this liquidity condition, but such a policy is not feasible as it requires negative dividends. From (14) it follows that

$$dDiv_t=rC_t\,dt-dC_t+dY_t. \tag{33}$$

The cash and dividend policy is feasible if the equality holds at each time. As the firm has full discretion over non-negative dividends, the cash policy remains feasible as long as $dDiv_t\geq 0$ in (33). The goal is to determine the lowest cash level $\overline{C}$ that satisfies both liquidity and feasibility conditions.

Suppose first that $\overline{C}(\mu,X)$ is a continuous and differentiable function. Applying Ito's lemma to $\overline{C}$, the right-hand side of (33) can be written as

$$dDiv_t=\left[r\overline{C}+(1-\tau)(\mu_t-k)-\frac{1}{2\sigma^2}(\mu_t-\mu_L)^2(\mu_H-\mu_t)^2\overline{C}_{\mu\mu}-\mu\overline{C}_X\right.$$
$$\left.-\frac{1}{2}\sigma^2\overline{C}_{XX}-(\mu_t-\mu_L)(\mu_H-\mu_t)\overline{C}_{\mu X}\right]dt$$
$$+\left[(1-\tau)\sigma-\frac{1}{\sigma}(\mu_t-\mu_L)(\mu_H-\mu_t)\overline{C}_{\mu}-\sigma\overline{C}_X\right]dZ_t, \quad (34)$$

where subindices at $\overline{C}$ denote partial derivatives. The requirement that increments of this process are non-negative for all $t\le t^*$ can be satisfied if and only if, first, the volatility coefficient at dZ_t is constant and zero and, second, the drift parameter at dt is non-negative. The first condition yields the following partial differential equation:

$$\frac{1}{\sigma^2}(\mu_t-\mu_L)(\mu_H-\mu_t)\overline{C}_{\mu}+\overline{C}_X=(1-\tau). \quad (35)$$

Its general solution is

$$\overline{C}(\mu,X)=(1-\tau)\frac{\sigma^2}{\mu_H-\mu_L}\ln\left(\frac{\mu-\mu_L}{\mu_H-\mu}\right)$$
$$+M_1\left[\frac{\mu_H-\mu_L}{\sigma^2}X-\ln\left(\frac{\mu-\mu_L}{\mu_H-\mu}\right)\right]+M_2, \quad (36)$$

where M_1 and M_2 are constants. As X_t, $t\le t^*$, can in general take any positive or negative values, the liquidity condition $C_t\ge 0$, $t\le t^*$, is satisfied only if M_1=0. This means that $\overline{C}$ is independent of X. To determine M_2, use the non-negativity condition on the drift parameter in (34), which, with the use of (36), can be written as

$$r\overline{C}(\mu,X)+(1-\tau)\left(\frac{\mu_H+\mu_L}{2}-k\right)\ge 0.$$

Note that $\overline{C}$ is increasing in μ, which implies that the inequality is most demanding at $\mu=\mu^*$. Moreover, the liquidity condition at all $t\le t^*$ requires that

$$\overline{C}(\mu^*,X)\ge 0.$$

Solving the last two inequalities for the constant M_2, one obtains the formula given in the proposition.

The final step is to rule out that there are points of discontinuity and non-differentiability in $\overline{C}$ if $\mu>\mu^*$. If $\overline{C}$ is discontinuous, it can only have downward jumps. But if immediately after the jump, $\overline{C}$ is the smallest C that allows the firm to avoid liquidity default, then in the continuous environment of the model, $\overline{C}$ before the jump could not be the smallest C satisfying this desired property. Hence, $\overline{C}$ must be continuous. Suppose now that $\overline{C}$ has some non-differentiable points. In between the points, $\overline{C}$ must satisfy differential equation (35) with the general solution in (36), subject to the boundary conditions implied by the continuity of $\overline{C}$. But with M_1=0, it will result in $\overline{C}$ that is a continuous differentiable function of μ for all $\mu>\mu^*$. □

Proof of Proposition 3. Define the time of liquidity default by $t_0=\inf\{t\ge 0: C_t<0\}$, the time of solvency default by $t^*=\inf\{t\ge 0:\mu_t<\mu^*\}$, and let $\tilde{t}=t_0\wedge t^*$. For a given μ^*, the equity value with the optimal dividend policy is given by

$$E(\mu,C)=\sup_{Div}\mathbb{E}_{\mu,C}\left[\int_0^{\tilde{t}}e^{-rs}dDiv_s+e^{-r\tilde{t}}C_{\tilde{t}}\right].$$

To shorten notation, it is useful to introduce the infinitesimal generator $\mathcal{A}$ of the two-dimensional process (μ,C) with dynamics described by (3) and (14). For a function $f(\mu,C)$ of class C^2, $\mathcal{A}$ is a partial differential operator describing the rate of change in f and by Ito's lemma is given by

$$\mathcal{A}f(\mu,C)=\frac{1}{2\sigma^2}(\mu-\mu_L)^2(\mu_H-\mu)^2f_{\mu\mu}(\mu,C)$$
$$+\frac{1}{2}(1-\tau)^2\sigma^2f_{CC}(\mu,C)$$
$$+(1-\tau)(\mu-\mu_L)(\mu_H-\mu)f_{\mu C}(\mu,C)$$
$$+[rC+(1-\tau)(\mu-k)]f_C(\mu,C).$$

I use the guessed dividend policy (19)–(21) to characterize $E(\mu,C)$ in different regions. For $\mu>\mu^*$ and $0<C<\overline{C}(\mu)$, $dDiv^*$=0 by (19) and, using Ito's lemma, $E(\mu,C)$ satisfies the differential equation:

$$rE(\mu,C)=\mathcal{A}E(\mu,C). \quad (37)$$

For $\mu\ge\mu^*$ and $C\ge\overline{C}(\mu)$, $E(\mu,C)$ is given by

$$E(\mu,C)=E_{nc}(\mu)+C. \quad (38)$$

For the case of $C=\overline{C}(\mu)$, (38) is proven in Proposition 4. The case $C>\overline{C}(\mu)$ follows then directly from (21). For $\mu\ge\mu^*$ and C=0, the firm defaults and $E(\mu,C)$ is

$$E(\mu,0)=0. \quad (39)$$

Note that while the guessed policy derives from the intuition gained from Proposition 2, in principle, a guess of the optimal dividend policy and equity value could have been found starting the variational inequality usual for a singular stochastic control problem of this type. Namely, one can expect that the optimal solution is a C^2 function that satisfies the variational inequality:

$$\max\{-rE(\mu,C)+\mathcal{A}E(\mu,C),1-E_C(\mu,C)\}=0,$$

and (39) at liquidity default.

The proof proceeds in two steps. First, it is proved that the policy specified in (19)–(21), Div^*, attains $E(\mu,C)$:

$$\mathbb{E}_{\mu,C}\left[\int_0^{\tilde{t}}e^{-rs}dDiv_s^*+e^{-r\tilde{t}}C_{\tilde{t}}\right]=E(\mu,C). \quad (40)$$

Second, it is shown that no other feasible dividend policy provides a higher value:

$$\mathbb{E}_{\mu,C}\left[\int_0^{\tilde{t}}e^{-rs}dDiv_s+e^{-r\tilde{t}}C_{\tilde{t}}\right]\le E(\mu,C). \quad (41)$$

In the first step, I start with the case of $0<C\leq\overline{C}(\mu)$. Applying Ito's lemma to $e^{-rt}E(\mu_t,C_t)$, it follows that

$$
\begin{aligned}
e^{-r(t\wedge\tilde{t})}E(\mu_{t\wedge\tilde{t}},C_{t\wedge\tilde{t}}) &= E(\mu,C)-\int_0^{t\wedge\tilde{t}} re^{-rs}E(\mu_s,C_s)\,ds\\
&+\int_0^{t\wedge\tilde{t}} e^{-rs}E_\mu(\mu_s,C_s)\,d\mu_s+\int_0^{t\wedge\tilde{t}} e^{-rs}E_C(\mu_s,C_s)\,dC_s\\
&+\int_0^{t\wedge\tilde{t}} e^{-rs}\frac{1}{2\sigma^2}(\mu_s-\mu_L)^2(\mu_H-\mu_s)^2E_{\mu\mu}(\mu_s,C_s)\,ds\\
&+\int_0^{t\wedge\tilde{t}} e^{-rs}\frac{1}{2}(1-\tau)^2\sigma^2E_{CC}(\mu_s,C_s)\,ds\\
&+\int_0^{t\wedge\tilde{t}} e^{-rs}(1-\tau)(\mu_s-\mu_L)(\mu_H-\mu_s)E_{\mu C}(\mu_s,C_s)\,ds.
\end{aligned}
$$

Moving $E(\mu,C)$ on the left-hand side and using the dynamics of μ and C results in

$$
\begin{aligned}
E(\mu,C) &= e^{-r(t\wedge\tilde{t})}E(\mu_{t\wedge\tilde{t}},C_{t\wedge\tilde{t}})\\
&-\int_0^{t\wedge\tilde{t}} e^{-rs}(-rE(\mu_s,C_s)+\mathcal{A}E(\mu_s,C_s))\,ds\\
&+\int_0^{t\wedge\tilde{t}} e^{-rs}E_C(\mu_s,C_s)\,dDiv^*_s-\int_0^{t\wedge\tilde{t}} e^{-rs}\\
&\times\left[\frac{1}{\sigma}(\mu_s-\mu_L)(\mu_H-\mu_s)E_\mu(\mu_s,C_s)+(1-\tau)\sigma E_C(\mu_s,C_s)\right]dZ_s.
\end{aligned}
$$

The first integrand on the left-hand side is equal to zero from (37) if $0<C<\overline{C}(\mu)$ or (38) combined with (5) if $C=\overline{C}(\mu)$. Given that the first derivatives of E are bounded, the last term is a martingale. Thus, taking expectations results in

$$E(\mu,C)=\mathbb{E}_{\mu,C}[e^{-r(t\wedge\tilde{t})}E(\mu_{t\wedge\tilde{t}},C_{t\wedge\tilde{t}})]+\mathbb{E}_{\mu,C}\left[\int_0^{t\wedge\tilde{t}} e^{-rs}E_C(\mu_s,C_s)\,dDiv^*_s\right].$$

Taking $t\to\infty$ leads to

$$E(\mu,C)=\mathbb{E}_{\mu,C}[e^{-r\tilde{t}}E(\mu_{\tilde{t}},C_{\tilde{t}})]+\mathbb{E}_{\mu,C}\left[\int_0^{\tilde{t}} e^{-rs}E_C(\mu_s,C_s)\,dDiv^*_s\right].$$

The first term on the left-hand side is equal to $\mathbb{E}_{\mu,C}[e^{-r\tilde{t}}C_{\tilde{t}}]$ by (38) and (39). As dividends are nonzero under Div^* only if $C_s=\overline{C}(\mu_s)$ and, by (38), $E_C(\mu,\overline{C}(\mu))=1$, the required equality (40) is satisfied.

Next, let $C>\overline{C}(\mu)$. In this case, Div^* and the corresponding process C are non-continuous at t=0. Using a generalized Ito's lemma to $e^{-rt}E(\mu_t,C_t)$ and setting $E(\mu,C)$ aside, one gets

$$
\begin{aligned}
E(\mu,C) &= e^{-r(t\wedge\tilde{t})}E(\mu_{t\wedge\tilde{t}},C_{t\wedge\tilde{t}})\\
&-\int_0^{t\wedge\tilde{t}} e^{-rs}(-rE(\mu_s,C_s)+\mathcal{A}E(\mu_s,C_s))\,ds\\
&+\int_0^{t\wedge\tilde{t}} e^{-rs}E_C(\mu_s,C_s)\,dDiv^*_s-\int_0^{t\wedge\tilde{t}} e^{-rs}\\
&\times\left[\frac{1}{\sigma}(\mu_s-\mu_L)(\mu_H-\mu_s)E_\mu(\mu_s,C_s)+(1-\tau)\sigma E_C(\mu_s,C_s)\right]dZ_s\\
&-E(\mu,\overline{C}(\mu))+E(\mu,C)+E_C(\mu,C)(\overline{C}(\mu)-C).
\end{aligned}
$$

Using (38) in the last term gives $-E(\mu,\overline{C}(\mu))+E(\mu,C)+E_C(\mu,C)(\overline{C}(\mu)-C)=-\overline{C}(\mu)+C+\overline{C}(\mu)-C=0$. Thus, following the same manipulations as in the previous case, one arrives at the assertion of equality (40).

The second step is to show that any other feasible dividend policy Div yields at most E. Let C be the cash holdings process corresponding to Div. As Div does not need to be continuous, one can decompose it to $Div_t=Div^c_t+\sum_{s\leq t}(Div_s-Div_{s-})$, where Div^c_t is the purely continuous part. Applying a generalized Ito's lemma to $e^{-rt}E(\mu_t,C_t)$ and rearranging to have $E(\mu,C)$ on the left-hand side leads to

$$
\begin{aligned}
E(\mu,C) &= e^{-r(t\wedge\tilde{t})}E(\mu_{t\wedge\tilde{t}},C_{t\wedge\tilde{t}})-\int_0^{t\wedge\tilde{t}} e^{-rs}(-rE(\mu_s,C_s)\\
&+\mathcal{A}E(\mu_s,C_s))\,ds+\int_0^{t\wedge\tilde{t}} e^{-rs}E_C(\mu_s,C_s)\,dDiv_s\\
&-\int_0^{t\wedge\tilde{t}} e^{-rs}\left[\frac{1}{\sigma}(\mu_s-\mu_L)(\mu_H-\mu_s)E_\mu(\mu_s,C_s)\right.\\
&\left.+(1-\tau)\sigma E_C(\mu_s,C_s)\right]dZ\\
&-\sum_{s\leq t\wedge\tilde{t}} e^{-rs}[E(\mu_s,C_s)-E(\mu_s,C_{s-})-E_C(\mu_s,C_{s-})(C_s-C_{s-})].
\end{aligned}
$$

The first integrand on the left-hand side is equal to zero from (37) if $0<C<\overline{C}(\mu)$ or (38) combined with (5) if $C=\overline{C}(\mu)$. As the first derivatives of E are bounded, the Ito integral on the left-hand side is a martingale. After taking expectations and rearranging the Div process, it follows that

$$
\begin{aligned}
E(\mu,C) &= \mathbb{E}_{\mu,C}[e^{-r(t\wedge\tilde{t})}E(\mu_{t\wedge\tilde{t}},C_{t\wedge\tilde{t}})]\\
&+\mathbb{E}_{\mu,C}\left[\int_0^{t\wedge\tilde{t}} e^{-rs}E_C(\mu_s,C_s)\,dDiv^c_s\right]\\
&-\mathbb{E}_{\mu,C}\left[\sum_{s\leq t\wedge\tilde{t}} e^{-rs}[E(\mu_s,C(\mu_s))-E(\mu_s,C_{s-})]\right].
\end{aligned}
$$

Observe that $E(\mu_s,C_s)-E(\mu_s,C_{s-})\leq C_s-C_{s-}$ as $E_C(\mu,C)\geq 1$. Moreover, $C_s-C_{s-}=-(Div_s-Div_{s-})$. It follows that

$$
\begin{aligned}
E(\mu,C) &\geq \mathbb{E}_{\mu,C}[e^{-r(t\wedge\tilde{t})}E(\mu_{t\wedge\tilde{t}},C_{t\wedge\tilde{t}})]\\
&+\mathbb{E}_{\mu,C}\left[\int_0^{t\wedge\tilde{t}} e^{-rs}E_C(\mu_s,C_s)\,dDiv^c_s\right]\\
&+\mathbb{E}_{\mu,C}\left[\sum_{s\leq t\wedge\tilde{t}} e^{-rs}(Div_s-Div_{s-})\right]\\
&\geq \mathbb{E}_{\mu,C}[e^{-r(t\wedge\tilde{t})}E(\mu_{t\wedge\tilde{t}},C_{t\wedge\tilde{t}})]+\mathbb{E}_{\mu,C}\left[\int_0^{t\wedge\tilde{t}} e^{-rs}\,dDiv^c_s\right]\\
&+\mathbb{E}_{\mu,C}\left[\sum_{s\leq t\wedge\tilde{t}} e^{-rs}(Div_s-Div_{s-})\right]\\
&\geq \mathbb{E}_{\mu,C}[e^{-r(t\wedge\tilde{t})}E(\mu_{t\wedge\tilde{t}},C_{t\wedge\tilde{t}})]+\mathbb{E}_{\mu,C}\left[\int_0^{t\wedge\tilde{t}} e^{-rs}(\mu_s,C_s)\,dDiv_s\right].
\end{aligned}
$$

Finally, taking limit $t\to\infty$, one reaches the required inequality (41). □

Proof of Proposition 4. Debt value is found as in the proof of Proposition 1. To determine equity value, one can use the general solution to differential equation (23). By direct verification, it is

$$E(\mu) = B_5(\mu-\mu_L)^{1-\beta}(\mu_H-\mu)^{\beta} + B_6(\mu-\mu_L)^{\beta}(\mu_H-\mu)^{1-\beta} + \frac{a_1}{r}\left[\ln\left(\frac{\mu-\mu_L}{\mu_H-\mu}\frac{\mu_H-\mu^*}{\mu^*-\mu_L}\right) + \frac{\mu_H-\mu_L}{r\sigma^2}\left(\mu-\frac{\mu_H+\mu_L}{2}\right)\right] + \frac{a_2}{r}.$$

Applying the boundary conditions at μ_H and μ^* to determine constants B_5 and B_6 leads to the expression provided in the proposition. □

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