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Finite project life and uncertainty effects on investment

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Abstract

This paper revisits the important result of the real options approach to investment under uncertainty, which states that increased uncertainty raises the value of waiting and thus decelerates investment. Typically in this literature projects are assumed to be perpetual. However, in today's economy firms face a fast-changing technology environment, implying that investment projects are usually considered to have a finite life. The present paper studies investment projects with finite project life, and we find that, in contrast with the existing theory, investments may be accelerated by increased uncertainty. It is shown that this particularly happens at low levels of uncertainty and when project life is short.

JEL classification: D92; E22; G31

Keywords: Investment; Uncertainty; Finite project length

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1. Introduction

The standard theory of the real options approach to investment, as clearly explained in Dixit and Pindyck (1994),[1] states that uncertainty in combination with irreversibility creates a value of the option to wait with undertaking capital investments. Over time more information becomes available, which enables the decision maker to make better investment decisions at a later date.

The general prediction of the real options literature is that a higher level of uncertainty increases the value of waiting and thus has a negative effect on investment. In this paper we revisit this conclusion. To do so we adopt the standard framework with contingent claims valuation of the investment opportunity and change it in one aspect: where the vast majority of the real options literature assumes projects to be perpetual, we allow for the project to generate earnings only during a finite amount of time.[2] The assumption of a project having an infinite life is useful mostly due to its simplicity. However, in corporate practice the investment projects are usually considered to have a finite life. This is especially true in today's knowledge economy, in which innovations limit the economic lifetime of technologies.[3] We show that the simplifying assumption of perpetual projects is critical for the investment–uncertainty relationship. Our main result is that the investment threshold decreases with uncertainty in case the uncertainty level is low and the project life is short. So, changing the project life from infinite to finite can imply a negative relationship between uncertainty and the value of waiting, which reverses the basic real options result.

To be more precise, an increase in uncertainty affects the investment threshold in three different ways. The first effect is the *discounting effect*. An increase of uncertainty raises the discount rate via the risk premium component. This reduces the net present value (NPV) of the investment and thus raises the investment threshold. The second effect is the *volatility effect*, which affects the value of the option to wait positively: higher uncertainty increases the upside potential payoff from the option, leaving the downside payoff unchanged at zero (since the option will not be exercised at low payoff values). This increased option value implies that the firm has more incentive to wait, which also increases the investment threshold. The third effect of an increase of uncertainty on the investment threshold is the *convenience yield effect*. The increase of asset riskiness raises the discount rate and thus also the convenience yield of the investment opportunity. This decreases the

[1] Some more recent contributions include studies of implications of learning (Decamps and Mariotti, 2004; Thijssen et al., 2006), agency (Grenadier and Wang, 2005), business cycle (Guo et al., 2005), policy change (Pawlina and Kort, 2005), and implications to capital structure choices (Miao, 2005), mergers and acquisitions dynamics (Morellec and Zhdanov, 2005), or exit strategies (Murto, 2004).

[2] Notably, Majd and Pindyck (1987) discuss some implications of finite project life on real options modeling. While they provide some arguments and cases when the finite project life considerations can be omitted, these considerations turn out to play an important role in our analysis.

[3] Certainly, the same arguments point toward introducing a finite life of the investment opportunity and not only of the project after investment. We do study this case in Section 4.1 where it is shown that our main result also holds there.

value of waiting, so that it is more attractive to invest earlier resulting in a lower investment threshold.

The discounting and volatility effects thus raise the investment threshold, while the convenience yield effect works in the opposite direction. Projects with a short life are relatively insensitive to discount rates. On the other hand, at low levels of uncertainty, increased uncertainty has still little effect on the probability of observing low prices, and thus the volatility effect is small in this case. Consequently, it is possible for the negative convenience yield effect to dominate the two other effects when the project life is finite and uncertainty is low. In that case it thus holds that the investment threshold decreases with uncertainty.

We examine the robustness of the non-monotonic effect of uncertainty on investment in the case of a finite project life by considering several variations of the problem. First, we show that this result survives in case the opportunity to invest in the project is available only for a limited amount of time. Next, we prove that this also holds for other relaxations of the infinite project life assumption, like uncertain project duration and capital depreciation. Furthermore, we find that generalized functional forms of the convenience yield preserve the observed relationships. Finally, the non-monotonic effect is also present in case revenues are mean reverting.

The impact of uncertainty on investments has been of interest to economists for a long time. One strand of literature relies on convex costs of capital adjustment and convexity of marginal profits in prices. As shown by Hartman (1972) and Abel (1983), in such a setting uncertainty hastens investment. The other important strand of literature, based on the real options theory, acknowledges (partial) irreversibility of investments and predicts that uncertainty delays investment. This paper verifies the latter prediction and shows that the investment trigger is not necessarily increasing in uncertainty. Most closely related papers are Caballero (1991) and Bar-Ilan and Strange (1996). Caballero (1991) considers a perfect competition setting with convex adjustment costs, and he obtains that irreversibility does not lead to the usual negative investment–uncertainty relationship. Bar-Ilan and Strange (1996) assume that there are lags between investment decisions and realizations. Firms have abilities to abandon uncompleted projects in bad times, which creates a convexity in the output and value functions. Bar-Ilan and Strange (1996) find that uncertainty may accelerate as well as decelerate investment depending on specific parameter values. Both papers have in common that they depart from the conventional result of the real options literature, because the models create convexities in line of Hartman (1972) and Abel (1983). Thus it comes with little surprise that in these papers uncertainty may either accelerate or decelerate investment. The result of our paper is unique in the sense that uncertainty may hasten irreversible investment without building on the convexity of the marginal product of capital. Our model remains in the pure real options framework and the reversal of the conventional result builds solely on the contingent claims valuation of investment opportunities and a finite capital lifetime. Moreover, since we only depart from the standard real option framework by imposing a finite lifetime, our model is more general and is thus applicable to more investment situations than Caballero (1991) and Bar-Ilan and Strange (1996).

A different approach to study the relationship between uncertainty and irreversible investments is taken by Sarkar (2000). Sarkar analyzes the probability of investment taking place within a certain time period and points at the fact that an increasing trigger does not automatically mean that investment will be delayed. A related result is shown by Ruffino and Treussard (2007) in a model of duopolistic competition with investment and time to build. The difference with our result is that we show that increased uncertainty may not even lead to an increased trigger.

Beyond this introduction the paper is organized as follows. In the next section we consider the model of the finitely lived project and derive the optimal investment trigger. Section 3 studies how uncertainty influences the investment decision. In Section 4 we discuss robustness, while Section 5 concludes. All proofs are contained in Appendix A.

2. The model and the optimal investment decision

We consider an irreversible investment project with finite life of T years that can be undertaken at any time. After the investment has taken place, the project generates a stochastic revenue of Q_t per unit time. Q_t evolves exogenously according to a geometric Brownian motion

$$\mathrm{d}Q_t = \mu Q_t \,\mathrm{d}t + \sigma Q_t \,\mathrm{d}Z_t, \tag{1}$$

where $\mathrm{d}Z$ is the increment of a standard Wiener process, μ is the drift parameter and σ is the volatility parameter that introduces the uncertainty in our model. Throughout the paper we assume that $\mu, \sigma > 0$. When the project is undertaken, a one-time investment cost I is paid. For simplicity, the marginal costs are put equal to zero.

We employ the contingent claims approach to real options valuation.[4] Under the standard assumption of market completeness, the expected rate of return of the project π is determined in the financial market equilibrium. The CAPM[5] formula relates π, the risk-free interest rate r, the correlation of the project return with the return of the market portfolio ρ, and the market price of risk λ as follows:

$$\pi = r + \lambda\rho\sigma. \tag{2}$$

The difference between π, the expected return of the project, and μ, the expected rate of change of Q, is referred to as the convenience yield (or return shortfall) of the investment opportunity. The later is denoted by δ and satisfies

$$\delta \equiv \pi - \mu = r + \lambda\rho\sigma - \mu. \tag{3}$$

[4]The standard methods in real options theory to value an investment opportunity are dynamic programming and contingent claims valuation (Dixit and Pindyck, 1994). Compared to dynamic programming, the contingent claims approach offers a better treatment of the discount rate, because it is endogenously determined as an implication of the overall equilibrium in capital markets.

[5]The assumption that the intertemporal Capital Asset Pricing Model (CAPM) of Merton (1973) holds is in accordance with the related literature. The CAPM brings a linear relationship between the discount rate and σ. In Section 4.3 we show that the results we present also hold for more general discount rate functions.

We assume that $\delta > 0$, which ensures that the investment is ever undertaken; otherwise it is never optimal to exercise the option.

The level of uncertainty faced by the firm is measured by the volatility parameter σ. From (3) we obtain that a change in σ results in a change of π, which must lead to an adjustment of either μ or δ or both. In general, this relation depends on what is assumed to be an endogenous parameter affected by changes in volatility. A certain guideline in this respect could be Pindyck (2004), that relates commodity inventories, spot and future prices and the level of volatility. The model is estimated for several commodities and the results show that a volatility shock has a significant effect on the convenience yield and only a small effect on the price. Consistent with this evidence, it also seems to be more common in the related literature on the investment–uncertainty relationship to assume that μ is fixed and δ changes with σ (e.g., Sarkar, 2000, 2003). We follow this modelling convention.

The value of the project $V(Q)$ evolves over time and depends on the current realization of Q. Upon installation the project value is equal to the expected present value of the revenue stream discounted by the risk-adjusted discount rate. If the project has a finite life of T years, then the project value at the time of the investment is

$$V(Q) = \mathrm{E}\left[\int_0^T \mathrm{e}^{-\pi t} Q_t \,\mathrm{d}t | Q_0 = Q\right] = \int_0^T \mathrm{e}^{-(\pi-\mu)t} Q \,\mathrm{d}t = Q\frac{1-\mathrm{e}^{-(r+\lambda\rho\sigma-\mu)T}}{r+\lambda\rho\sigma-\mu}. \tag{4}$$

Before the project is installed, the firm holds an option to invest. The option is held until the stochastic revenue flow reaches a sufficiently high level at which it is optimal to exercise the option and invest. The option value $F(Q)$ can be found by the replicating portfolio argument. Employing the standard methods (cf. Dixit and Pindyck, 1994) yields that $F(Q)$ must satisfy the differential equation:

$$\tfrac{1}{2}\sigma^2 Q^2 F''(Q) + (\mu - \lambda\rho\sigma) Q F'(Q) - rF(Q) = 0. \tag{5}$$

We solve this differential equation subject to the value matching and smooth pasting conditions at the investment trigger Q^* and a zero value condition at $Q = 0$. The derivations are standard and are omitted here. The resulting firm value prior to investment is

$$F(Q) = (V(Q^*) - I)\left(\frac{Q}{Q^*}\right)^{\beta_1}.$$

The optimal investment rule is given by the upper trigger

$$Q^* = \frac{\beta_1}{\beta_1 - 1}\frac{r+\lambda\rho\sigma-\mu}{1-\mathrm{e}^{-(r+\lambda\rho\sigma-\mu)T}} I, \tag{6}$$

while β_1 is the positive root of the quadratic equation

$$L_0 \equiv \tfrac{1}{2}\sigma^2 \beta(\beta - 1) + (\mu - \lambda\rho\sigma)\beta - r = 0 \tag{7}$$

and equals

$$\beta_1 = \frac{1}{2} - \frac{\mu - \lambda\rho\sigma}{\sigma^2} + \sqrt{\left(\frac{\mu - \lambda\rho\sigma}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}, \tag{8}$$

Under the NPV rule the investment is undertaken as soon as the risk-adjusted project value exceeds the investment cost, that is at the revenue level equal to $\frac{r+\lambda\rho\sigma-\mu}{1-e^{-(r+\lambda\rho\sigma-\mu)T}}I$. This value is always lower than Q^* in (6), as $\beta_1 > 1$. So there are states where the expected payoff of investment is positive and the firm chooses to wait and not to invest. The option to invest captures this positive value of waiting.

3. The effects of uncertainty on the investment trigger

This section studies the effect of uncertainty on the value of waiting. First, we show that, as usual, the value of waiting, reflected in the level of investment trigger, always increases with uncertainty when the project life is infinite or when discount rates are unaffected by uncertainty. Second, if the equilibrium discount rate contains a positive risk premium, we derive that the value of waiting decreases with uncertainty in case of finite project lives and low levels of uncertainty. Finally, we provide an economic analysis of these results.

3.1. Monotonicity results

We start out with the basic real options result for the investment project with infinite life.

Proposition 1. *If the project life is infinite, the investment trigger increases with uncertainty.*

In case of an infinite project life the effect of uncertainty on the investment trigger is unambiguously positive. This is the standard real options result, which says that the value of waiting increases with uncertainty. This is reflected by higher trigger values, because the revenue must reach a higher level before investment is optimally undertaken.

Now, let us move on to the finite life project case. We first consider the scenario where the impact of systematic risk is absent or not priced by the market. This implies that the discount rate is constant, and requires that either the market price of risk is zero, $\lambda = 0$, or that the correlation of the project return with the return of the market portfolio is zero, $\rho = 0$.

Proposition 2. *If the discount rate is constant (zero market price of risk or zero correlation of project return with the return of the market portfolio), the relationship between the investment trigger and uncertainty is always positive.*

Proposition 2 states that, in the absence of the risk premium effect the investment trigger always increases with uncertainty irrespective of the project lifetime, which is again the usual real options result. It is important to point out, however, that the conditions necessary for constant discount rates are in general difficult to accept in the context of investment models; see discussions in, e.g., Zeira (1990) and Sarkar (2003).

The next proposition considers one case where the discount rate is not constant.

Proposition 3. *If $\lambda\rho < 0$, then the relationship between the investment trigger and uncertainty is always positive.*

This result shows that in case of a negative risk premium (possible if either the correlation of the project return with the return of the market portfolio or the market price of risk is negative), the usual positive relationship arises.

3.2. Non-monotonicity result

We proved in the previous subsection that both in the model with a project of infinite life and in the model without a risk premium or with a negative risk premium, the impact of uncertainty on the investment trigger is always positive. These are interesting special and limit cases; however, the assumptions of Propositions 1 and 2 are serious abstractions from reality, and the negative risk premium condition of Proposition 3 is a relatively rare phenomenon in the markets. Next, we turn to the most common situation where the project life is finite and the discount rate is set in the capital market equilibrium with a positive risk premium. We now show that the effect on the trigger is no longer monotonic in uncertainty.

Proposition 4. *If the project life is finite and $\lambda\rho > 0$, the uncertainty effect on the investment trigger is non-monotonic: it decreases in σ for low levels of σ and then increases. The length of the σ-interval where the negative effect occurs is negatively related to the project lifetime.*

Fig. 1 presents some numerical examples, where the parameter values correspond to earlier work on the investment–uncertainty relationship, in particular to Sarkar (2000). We see that indeed there is a negative relation between σ and Q^* for lower values of σ. The effect is more pronounced for short-term projects, but even in the case of a 30-year project Q^* decreases until σ is around 0.12. The example shows that the positive effect of uncertainty on investment (negative on the trigger) arises for economically relevant parameter values. The figure, of course, also confirms that for an infinitely long project the relation is monotonic and increasing in line of the results in Proposition 1.

3.3. Economic analysis of the non-monotonicity result

In this section we provide an economic interpretation of the non-monotonic effect of uncertainty shown in Proposition 4 (we assume here that $\lambda\rho > 0$). From (3) and (6) it follows that the investment trigger can also be expressed as

$$Q^* = \frac{\beta_1}{\beta_1 - 1} \frac{\delta}{1 - e^{-\delta T}} I. \tag{9}$$

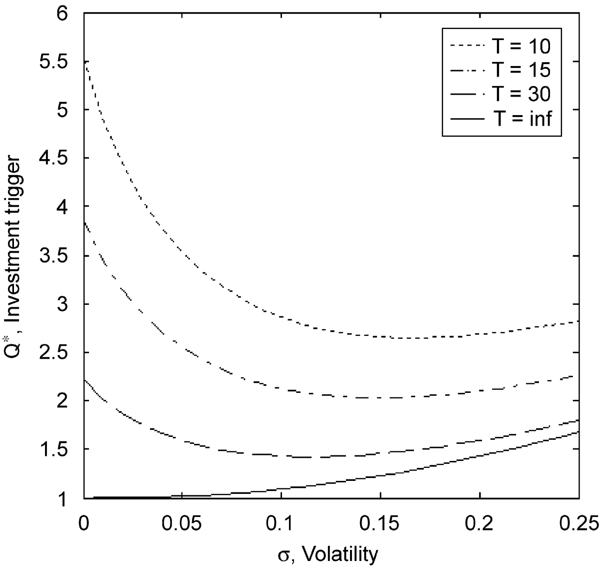

Fig. 1. Investment trigger as a function of volatility for various project lengths T and the set of parameters: $\mu = 0.08$, $r = 0.1$, $\rho = 0.7$, $\lambda = 0.4$, $I = 10$.

At this point it is convenient to trace all the variables that are affected by uncertainty and consider the trigger as a function of three parameters: $Q^*(\sigma, \delta(\sigma), \beta_1(\sigma, \delta(\sigma)))$. Then the derivative of the investment trigger with respect to σ can be decomposed into three effects in the following way:

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} Q^*(\sigma, \delta(\sigma), \beta_1(\sigma, \delta(\sigma))) = \underbrace{\frac{\partial Q^*}{\partial \delta}\frac{\partial \delta}{\partial \sigma}}_{\text{Discounting effect}} + \underbrace{\frac{\partial Q^*}{\partial \beta_1}\frac{\partial \beta_1}{\partial \sigma}}_{\text{Volatility effect}} + \underbrace{\frac{\partial Q^*}{\partial \beta_1}\frac{\partial \beta_1}{\partial \delta}\frac{\partial \delta}{\partial \sigma}}_{\text{Convenience yield effect}}. \tag{10}$$

The three effects have a clear interpretation and each has an unambiguous sign (for the case of $\lambda\rho > 0$). The first term on the right-hand side, the *discounting effect*, measures the impact of revenue uncertainty on the rate used to discount the project value. Rising uncertainty increases the discount rate, which reduces the NPV of the investment project. This implies that it is less profitable to invest in this project, which leads to an increase of the trigger value. Consequently, as is straightforward to derive, the discounting effect is always positive.

Since the second and the third term of (10) both affect the trigger value via β_1, they reflect the influence of uncertainty on the value of the option to wait. Below we refer to these two effects combined as the *option effect*. The *volatility effect*, which is represented by the derivative $\frac{\partial Q^*}{\partial \beta_1}\frac{\partial \beta_1}{\partial \sigma}$, captures the direct impact of uncertainty on the value of the option to wait. Higher uncertainty increases the upside potential payoff from the option, leaving the downside payoff unchanged at zero (since the option will not be exercised at low payoff values). This is the well-known positive impact of uncertainty on the option value. An increased option value implies that the firm has more incentive to wait. This raises the opportunity cost of investing so that the investment trigger will increase. Hence, the effect is unequivocally positive.

The product $\frac{\partial Q^*}{\partial \beta_1}\frac{\partial \beta_1}{\partial \delta}\frac{\partial \delta}{\partial \sigma}$ in (10) represents the impact of uncertainty on the option value through the convenience yield, and we refer to it as the *convenience yield effect*. Increased uncertainty raises the risk premium of the expected rate of return and thus also the convenience yield, which in turn elevates the opportunity cost of holding the option and consequently decreases its value. For this reason it is attractive to invest earlier, which reduces the trigger.

Summarizing, we conclude that the discounting and volatility effects are positive, while the convenience yield effect is negative. Clearly, the negative relationship between uncertainty and investment occurs only if the convenience effect dominates the two other effects. The following proposition shows how the uncertainty level and the project length influence the relative size of the three effects.

Proposition 5. (i) *Define* $\hat{\sigma} = \{\sigma \geqslant 0 : (\beta_1 - 1)\sigma - \lambda\rho = 0\}$. *The combined option effect is negative at* $\sigma < \hat{\sigma}$ *and positive at* $\sigma > \hat{\sigma}$.

(ii) *The shorter is the project life* T, *the smaller is the discounting effect and the larger in absolute terms are the two option effects.*

The first part of the proposition states that the sign of the effect of uncertainty on the option value is ambiguous but separable into two regions.[6] At a relatively high uncertainty level the positive volatility effect dominates the negative convenience yield effect. At low levels of uncertainty the negative effect dominates. In such a case, a marginal increase in uncertainty has little impact on the probability of reaching extreme values by the underlying process and hence the volatility effect is relatively small. On the other hand, the convenience yield effect is also significantly present at low levels of uncertainty, since the convenience yield δ is linear in σ, implying that the marginal effect of σ in δ is constant (in fact the convenience yield effect is not constant but diminishes at higher σ, as the full effect works via the discount factor).

The second part of the proposition states that the project and option-related effects react differently to changes in the project life. The discounting effect becomes smaller with shorter project lives. Clearly, short-lived projects are relatively insensitive to marginal changes of the discount rate. On the other hand, the option-related effects increase with shorter project lives. This is because a shorter project life implies that the current revenue flow needs to be larger for the investment to be optimal, which leads to larger option effects.

Now we are ready to establish when and why an increasing uncertainty level may lower the investment threshold. At low levels of uncertainty, the positive volatility

[6]From Proposition 5 (i) it is clear that in a setup where only the option effects are present, the non-monotonic investment–uncertainty relationship would arise irrespective of the project lifetime. This could be the case for example, if the project value V behaves according to geometric Brownian process. This was shown in a contemporaneous work by Wong (2007). However such a setup is a rather serious abstraction from reality (see Dixit and Pindyck, 1994, p. 175 for arguments) and the negative effect disappears as soon as perpetual revenues from the project are directly modelled.

Table 1
The three effects of uncertainty affecting the position of the investment trigger for the set of parameters: $\mu = 0.08$, $r = 0.1$, $\rho = 0.7$, $\lambda = 0.4$, $I = 10$, $Q = 1$

	$T = 10$					$T = 30$				
σ	Q^*	(1)	(2)	(3)	(4)	Q^*	(1)	(2)	(3)	(4)
0.00	5.52	7.47	0.00	−77.23	−69.77	2.22	8.38	0.00	−31.03	−22.65
0.02	4.44	5.95	1.19	−48.51	−41.37	1.87	6.86	0.50	−20.44	−13.08
0.04	3.77	5.01	2.15	−33.70	−26.54	1.66	5.92	0.95	−14.86	−8.00
0.06	3.34	4.39	2.99	−25.03	−17.65	1.54	5.29	1.38	−11.53	−4.86
0.08	3.05	3.97	3.75	−19.47	−11.75	1.46	4.88	1.80	−9.35	−2.67
0.10	2.86	3.68	4.40	−15.66	−7.57	1.43	4.60	2.20	−7.82	−1.02
0.12	2.74	3.49	4.95	−12.92	−4.48	1.42	4.43	2.57	−6.70	0.29
0.14	2.67	3.38	5.36	−10.90	−2.17	1.44	4.34	2.88	−5.86	1.36
0.16	2.65	3.31	5.65	−9.38	−0.42	1.47	4.30	3.15	−5.22	2.22
0.18	2.65	3.28	5.85	−8.24	0.90	1.53	4.30	3.37	−4.74	2.93
0.20	2.68	3.28	5.99	−7.37	1.91	1.59	4.34	3.55	−4.37	3.52
0.22	2.73	3.31	6.09	−6.70	2.69	1.67	4.39	3.72	−4.09	4.02
0.24	2.79	3.34	6.17	−6.19	3.32	1.75	4.46	3.87	−3.89	4.45

The columns present: the discounting effect (1), the volatility effect (2), the convenience yield effect (3) and the total effect (4).

effect is small and the effects working via discount rate and convenience yield are still significant. These two last effects have opposing signs so that a low σ alone is not enough to observe a negative total effect (cf. Proposition 1). If, however, in addition the project life is short then the positive discounting effect will be small and the negative convenience yield effect dominates. Therefore, at low levels of σ and T, it is possible that the negative convenience yield effect dominates the two positive effects (see Proposition 4).

These mechanisms are illustrated in a numerical example presented in Table 1. It allows for a closer inspection of the magnitude of the effects of uncertainty affecting the position of the investment trigger. The volatility and convenience yield effects increase with shortening the project life. The discounting effect decreases with smaller T. The combined option effect is negative for low levels of σ but it is increasing in σ (it becomes positive for $\sigma > \hat{\sigma} = 0.241$). The longer the project life, the faster is the negative convenience yield effect offset by the positive impact of the discounting and volatility effects. If $T = 10$, the total effect is negative for σ between 0 and 0.16, while for $T = 30$ the total effect remains negative for σ between 0 and 0.10.

4. Robustness

The model of the previous sections has been geared to show our results in the simplest setting. The aim of this section is to demonstrate that our main result, i.e. that the value of waiting decreases with uncertainty in case of a short project life and a limited amount of uncertainty, can be generalized. First we consider a scenario where the investment opportunity is available only for a limited amount of time. After that we analyze the case where the project has an uncertain duration. Next, we consider more general, thus not necessary linear, convenience yield functions in uncertainty. Finally, we allow the revenue process to be mean reverting.

4.1. Finite-life option

We now assume that the project and the option to invest both have finite durability. McDonald and Siegel (1986) also allow for a finite life of the investment opportunity, but their project is implicitly perpetual. Finite life options have been extensively studied, and the book by Detemple (2005) provides background on recent analytical, approximation and numerical methods. A new numerical approach has been recently proposed by Nagae and Akamatsu (2007), in which the real options problem is reformulated as a system of complementarity problems.

The project life is T years and its value $V(Q)$ is given by Eq. (4). Denote the life length of the option as T_F. Since the option expires at T_F, its value $F(Q, \tau_F)$ depends on remaining time τ_F to maturity. To find the differential equation defining the option value we follow the same steps as in Section 2. The resulting partial differential equation includes the time derivative and is given by

$$\tfrac{1}{2}\sigma^2 Q^2 F_{QQ} + (\mu - \lambda\rho\sigma)QF_Q - F_\tau - rF = 0. \tag{11}$$

The option value must satisfy the terminal condition at the expiry date T_F :

$$F(Q,0) = \max(V(Q) - I, 0),$$

which states that at $\tau = 0$ the option is exercised (the investment is undertaken) if the project's expected present value exceeds the investment cost. The option satisfies also the boundary conditions at $Q = 0$ and Q^* similar to the ones used in Section 2: $F(Q^*, \tau_F) = V(Q^*) - I$, $F_Q(Q^*, \tau_F) = V'(Q^*)$ and $F(0, \tau_F) = 0$.

Unlike in the previous problem, in which Q^* was a single point, here the optimal investment trigger $Q^*(\tau_F)$ is a function of time. The problem we have to solve is analogous to the valuation of American-style options with a finite expiry date, to which no closed-form solutions exist. We numerically solve Eq. (11) together with the boundary conditions using the Crank–Nicholson finite-difference scheme. We apply the logarithmic change of variable and use a mesh size of 500×500 points. The computer code is available through the JEDC Supplement Archive.

Figs. 2 and 3 present our results for the optimal investment trigger boundary $Q^*(\tau_F)$. We assumed the option life T_F to be 10 years and the project life T to be either 10 years (Fig. 2), or perpetual (Fig. 3). All other parameters are as in the numerical example of Fig. 1. The triggers $Q^*(\tau_F)$ are drawn for various levels of σ ranging from 0.10 to 0.30. The horizontal axis depicts the remaining option life τ_F.

As expected, the right-hand side of both figures at $\tau_F = T_F = 10$ is well approximated by the model with a perpetual real option, so that the trigger boundary values are very close to those in Fig. 1 ($T = 10$ and ∞ curves). At $\tau_F = 0$,

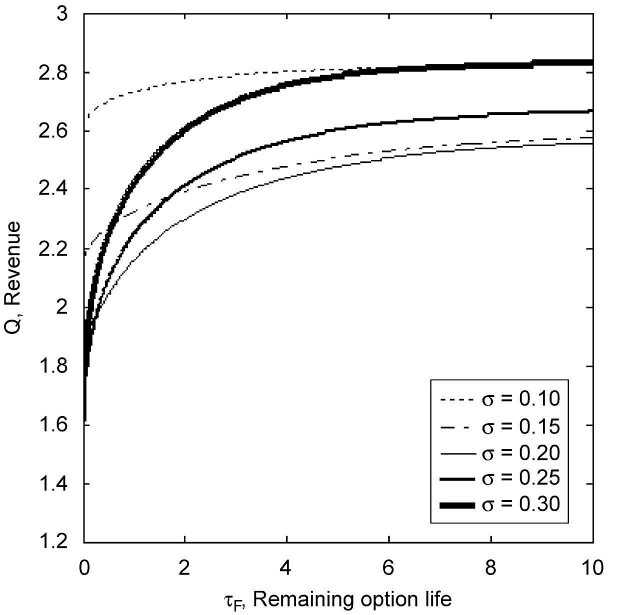

Fig. 2. Project and option with finite life: investment trigger boundary, $Q^*(\tau_F)$, for various levels of volatility and the set of parameters: $\mu = 0.08$, $r = 0.1$, $\rho = 0.7$, $\lambda = 0.4$, $I = 10$, $T = 10$, $T_F = 10$.

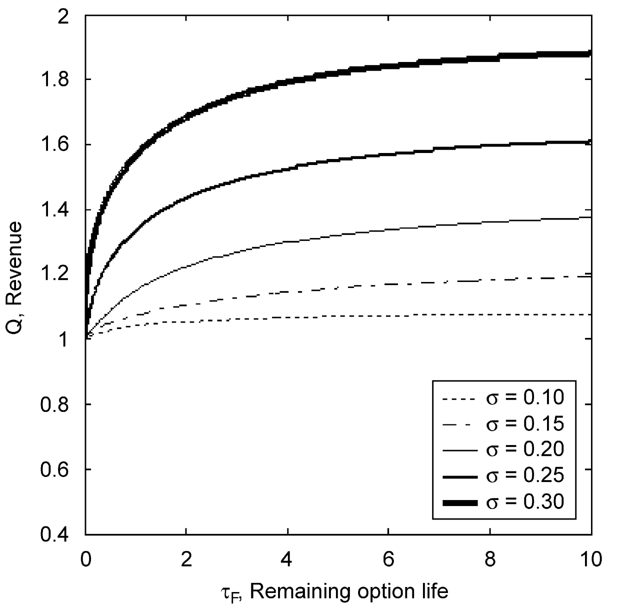

Fig. 3. A perpetual project and a finitely lived option to invest: investment trigger boundary, $Q^*(\tau_F)$ for various levels of volatility and the set of parameters: $\mu = 0.08$, $r = 0.1$, $\rho = 0.7$, $\lambda = 0.4$, $I = 10$, $T = \infty$, $T_F = 10$.

when the investment decision becomes a now-or-never decision, all curves are at the values implied by the NPV investment rule.

Fig. 2 clearly confirms our result that a finite project life may cause the real option investment rule to be non-monotonic in uncertainty. An increase of σ from 0.10 to 0.15 moves the curve downwards.[7] But an increase of σ from 0.20 to 0.25 and 0.30 shifts the optimal triggers upwards. The important finding of this numerical analysis is that after comparing Figs. 2 and 3, we can conclude that the levels of σ at which the trigger decreases and increases with uncertainty, remain roughly the same. In both cases the revenue uncertainty level at which the change of sign occurs lies between $\sigma = 0.15$ and 0.20. Thus the finite-life option assumption neither mitigates nor augments the positive relationship between investment and uncertainty due to the decreasing trigger.

Fig. 2 shows also that the effect of uncertainty may differ depending on the remaining option life. The dashed curve of $\sigma = 0.15$ is below the dot-marked curve of $\sigma = 0.25$ at high τ_F and above at low τ_F. The reason is the nearly flat horizontal shape of the optimal investment trigger curve at relatively low σ ($\sigma = 0.10$ or 0.15) for most of the option life and a sudden drop close to $\tau_F = 0$. This shape is caused by the convenience yield being low at lower σ, implying that there is only a small gain of undertaking the investment early (recall that a call option is never prematurely exercised if the convenience (dividend) yield is zero).

The behavior of the investment boundary in Fig. 2 can be contrasted with the case of the perpetual project. Fig. 3 shows that when the project life is infinite then $Q^*(t)$ moves upwards with increasing uncertainty. This is the usual monotonic relation consistent with the model with perpetual opportunity to invest.

4.2. Stochastic project life

An alternative for assuming a deterministic finite project life is to impose that a Poisson arrival brings the project to an end. We study this here and assume that the project lifetime (after installation) follows a Poisson process with rate γ. Among the numerous studies applying this setup we like to mention Merton (1976), who uses it in a financial option context, and McDonald and Siegel (1986), who apply it to the case of real investments.

Using Eq. (4) and the probability density of the stochastic lifetime, we obtain the project value

$$V(Q) = \int_0^\infty Q \frac{1 - \mathrm{e}^{-(r+\lambda\rho\sigma-\mu)t}}{r + \lambda\rho\sigma - \mu} \gamma \, \mathrm{e}^{-\gamma t} \, \mathrm{d}t = \frac{Q}{r + \lambda\rho\sigma - \mu + \gamma}.$$

Note that the mortality rate γ leads to an environment equivalent to the one with perpetual projects except that the effective discount rate is now $r + \lambda\rho\sigma - \mu + \gamma$ rather than $r + \lambda\rho\sigma - \mu$. The resulting formula prompts that a project with stochastic lifetime can be interpreted as a perpetual project that is exponentially depreciated with rate γ (see Dixit and Pindyck, 1994, p. 200).

[7] Except at the expiry date $\tau_F = 0$, at which $Q^*(t)$ increases in σ for all σ.

Analogous to the previous analyses, the optimal investment trigger can be derived as follows:

$$Q^{*} = \frac{\beta_1}{\beta_1 - 1}(r + \lambda\rho\sigma - \mu + \gamma)I. \tag{12}$$

We can now show that the non-monotonic uncertainty effect carries over to the case of a stochastic project life.

Proposition 6. *If $\gamma > 0$ and $\lambda\rho > 0$, then the uncertainty effect on the investment trigger is non-monotonic: it decreases in σ for low levels of σ and then increases. The length of the σ-interval where the negative effect occurs increases in γ.*

This result points out how strongly the monotonic relationship between the investment trigger and uncertainty hinges on the assumption of the project being perpetual. If there exists just a small probability that the project will be finished in finite time, the investment trigger will be decreasing with increasing uncertainty for a small enough σ. To illustrate this result, a numerical example is presented in Fig. 4. Here we indeed see that even a very small γ causes the trigger to decrease in uncertainty at low but realistic levels of uncertainty. We also see that the boundary moves upward as γ increases, reflecting that a higher instantaneous flow of is needed for the investment to be optimal, once the probability that a project ends increases.

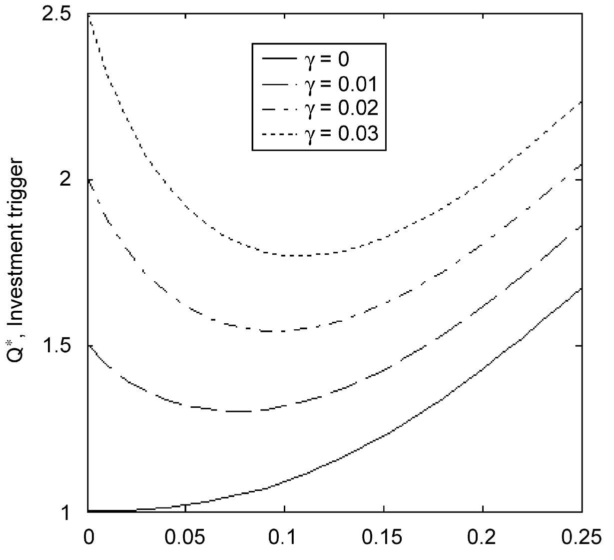

Fig. 4. Investment trigger as a function of volatility for various Poisson arrival rates γ and the set of parameters: $\mu = 0.08$, $r = 0.1$, $\rho = 0.7$, $\lambda = 0.4$, $I = 10$.

4.3. General convenience yield

The previous results stated in Propositions 1–6 are obtained for the framework of Section 2 (and Section 4.2 in the stochastic life case). In that model, the equilibrium discount rate, and also the convenience yield, are determined by the standard CAPM and thus are linear in σ. Here we check whether this linearity is crucial for the results that we obtained. This issue is relevant as, apart from the standard CAPM, there exist theory and some evidence in favor of nonlinearity. For example, it is well known that the presence of finite heterogeneous investment horizons leads to a nonlinear CAPM with a nonlinear relationship between returns and risk (see e.g., Lee et al., 1990). Moreover, there is a growing literature on factor pricing models with nonlinearities (see Bansal and Viswanathan, 1993).

Let the convenience yield be a non-decreasing, continuous, twice differentiable function of uncertainty $\delta(\sigma)$ for $\sigma \geqslant 0$. In the previous sections we obtained results for the linear case, i.e. $\delta''(\sigma) = 0$. We now present propositions that generalize those results. Corresponding to Proposition 1 we have that

Proposition 7. *If the project life is infinite and* $\delta'(\sigma) \geqslant 0$, *then the investment trigger increases with uncertainty.*

Proposition 4 can be generalized as follows.

Proposition 8. *If the project life is finite,* $\delta'(\sigma) > 0$ *and* $\delta''(\sigma) \leqslant 0$, *then the uncertainty effect on the investment trigger is non-monotonic: it decreases in* σ *for low levels of* σ *and then increases. The length of the* σ*-interval where the negative effect occurs decreases with project lifetime.*

So in the case of a finite project life, the previously observed properties for linear $\delta(\sigma)$ carry over to a concave $\delta(\sigma)$. In case of a convex $\delta(\sigma)$, we can have either a U-shaped relationship and a monotonic one.[8]

4.4. Mean reverting revenues

In this section we relax the assumption that revenue follows a geometric Brownian motion by allowing Q to be mean reverting. There have been several studies that considered the impact of mean revision on real options valuation (Metcalf and Hassett, 1995; Schwartz, 1997; Sarkar, 2003). We analyze here whether our result that a finite project life may cause a non-monotonic investment–uncertainty relationship carries over to the framework with mean revision.

Suppose that the revenue flow follows a geometric mean reverting process characterized by the following stochastic differential equation:

$$\mathrm{d}Q_t = [\mu Q_t + \kappa(\theta\, \mathrm{e}^{\mu t} - Q_t)]\,\mathrm{d}t + \sigma Q_t\, \mathrm{d}Z_t. \tag{13}$$

[8] To check it, take, for instance, $\delta(\sigma) = r + \lambda\rho\sigma^{3/2} - \mu$ with the parameter values as in Table 1 and the uncertainty effect is U-shaped. However, if $\delta(\sigma) = r + \lambda\rho\sigma^3 - \mu$, the effect of uncertainty is always positive.

The process corresponds to the generalized mean revision in Eq. (2) of Metcalf and Hassett, 1995. $\kappa > 0$ is the speed of revision of the process towards its mean. The mean is $\theta\,e^{\mu t}$ and grows exponentially at rate $\mu > 0$. If $\kappa = 0$ the process becomes a geometric Brownian motion with drift μ as in (1). If $\mu = 0$, the process in (13) becomes a simple mean revision with constant mean as studied by Sarkar (2003).

Denote the project value with remaining time τ to maturity at time t by $V(Q, \tau, t)$ (the mean of Q depends on calendar time and this dependence is reflected in V). Using standard arguments, we find that $V(Q, \tau, t)$ must satisfy the following differential equation:

$$\tfrac{1}{2}\sigma^2 Q^2 V_{QQ} + [(\mu - \lambda\rho\sigma)Q + \kappa(\theta\,e^{\mu t} - Q)]V_Q - V_\tau - rV + Q = 0 \tag{14}$$

with the terminal condition at maturity $\tau = 0$,

$$V(Q, 0, t) = 0. \tag{15}$$

Differential equation (14) with boundary condition (15) has an analytical solution[9] which is linear in Q:

$$V(Q, T, t) = AQ + B,$$

where

$$A = \frac{1 - e^{-(r+\lambda\rho\sigma-\mu+\kappa)T}}{r + \lambda\rho\sigma - \mu + \kappa},$$

$$B = \frac{\kappa\theta\,e^{\mu t}}{\kappa + \lambda\rho\sigma}\left[\frac{1 - e^{-(r-\mu)T}}{r - \mu} - \frac{1 - e^{-(r+\lambda\rho\sigma-\mu+\kappa)T}}{r + \lambda\rho\sigma - \mu + \kappa}\right].$$

As expected, when $\kappa = 0$ the value function is identical to (4) with revenues following a geometric Brownian motion. When $\mu = 0$ the formula simplifies to the value function in Eq. (2) in Sarkar (2003).

Similarly, using standard arguments one can show that the value of the option to invest $F(Q)$ satisfies

$$\tfrac{1}{2}\sigma^2 Q^2 F_{QQ} + [(\mu - \lambda\rho\sigma)Q + \kappa(\theta\,e^{\mu t} - Q)]F_Q - rF = 0 \tag{16}$$

with boundary conditions: $F(Q^{*}) = AQ + B - I$, $F_Q(Q^{*}) = A$ and $F(0) = 0$. The differential equation (16) with the boundary conditions has no known analytical solution, but it can be readily solved numerically. To find the optimal investment trigger we use a simple shooting method. The method is very accurate as long as the value function does not have to be evaluated numerically (see Dangl and Wirl, 2003 for more details and further discussion). We convert the second order differential equation (16) into a system of two first order differential equations and employ a Runge–Kutta algorithm to solve the initial value problem. The computer code is available through the JEDC Supplement Archive.

[9] The analytical solution for the project value with finite lifetime when revenues follow a generalized geometric mean reverting process (13) might be of interest on its own; see also Li (2003) who solves a similar problem.

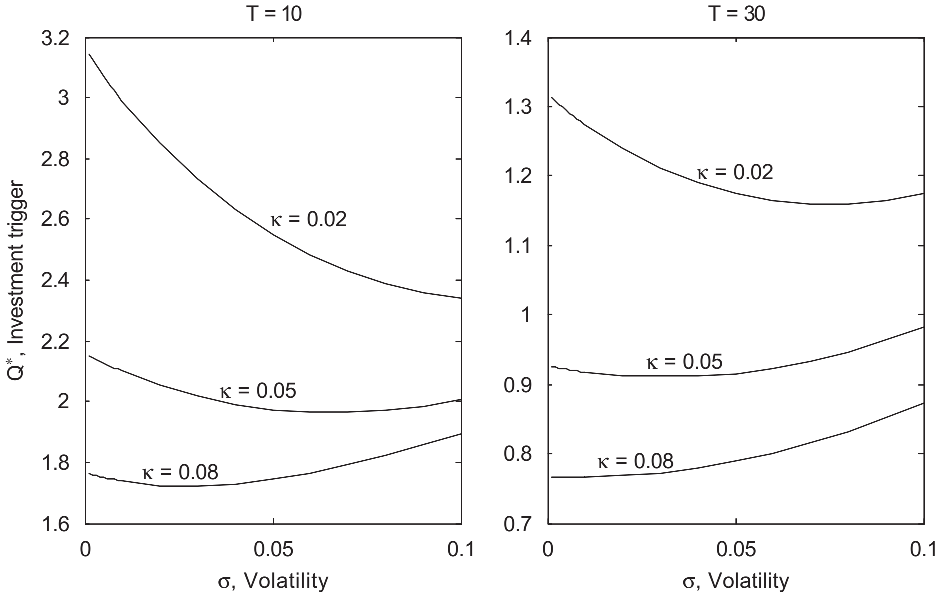

Fig. 5. Investment trigger as a function of volatility for various levels of speed of revision κ and the set of parameters: $\mu = 0.08$, $r = 0.1$, $\rho = 0.7$, $\lambda = 0.4$, $\theta = 0.5$, $I = 10$, $t = 0$. Project lifetime T is 10 (left) and 30 (right).

To examine the effect of uncertainty on investment in the presence of mean revision, we repeat the numerical exercise for various project durations and levels of speed of revision κ. Fig. 5 illustrates the results for two different project lifetimes $T = 10$ and 30, and various levels of κ. The other parameters are as in the previous numerical examples with the addition of $\theta = 0.5$ and $t = 0$. In principle, for each t the mean of Q is different (it grows deterministically and equals $\theta\,\mathrm{e}^{\mu t}$) and so the trigger strategy changes over time. θ is the mean at $t = 0$ and its value is chosen in such a way that it is not above the optimal investment triggers at $t = 0$.

It is clear that in general uncertainty effects are less pronounced in the presence of mean revision. Therefore, as illustrated in Fig. 5 uncertainty effects on investment are flattened especially for larger κ and long-lived projects. Yet the main result of this paper still holds, since the non-monotonic relationship between uncertainty and investment is present if the project life is short and the region of the negative effect is larger the shorter is the project lifetime. For higher levels of κ and for larger T the uncertainty effect weakens and ultimately the effect only holds for very low values of σ.

5. Conclusions

Our paper shows that a finite life of an investment project in combination with a risk premium in expected rates of return may reverse the usual effect of uncertainty on irreversible investments. In particular, we determined a scenario under which increased uncertainty reduces the value of waiting with investment. We now briefly discuss some implications of this result.

In corporate practice investment projects are usually considered to have a finite life, which supports the importance of our result. It thus seems that assuming the project life to be infinite, which is done in the overwhelming majority of real options contributions, is useful for simplicity reasons but dangerous since adverse uncertainty effects are lost.

From a policy point of view our results demonstrate that there exists a positive level of uncertainty at which the investment trigger admits its lowest value. If the policy aim is to increase investment, then the implication is that it is not necessarily optimal in all cases to decrease the level of uncertainty of policy instruments. However, any specific recommendation may be a bit far-reaching in the current single-firm model with a general source of uncertainty. To derive policy implications out of our non-monotonic investment–uncertainty relationship deserves a separate study. Similarly, in order to focus on the main features of the described mechanism, we have not attempted to construct a richer model of industry equilibrium. This can be done by considering a competitive industry (as in Caballero and Pindyck, 1996 and others) or imperfect competition (as in Smets, 1991; Grenadier, 1996 and others). However, we are quite confident that, qualitatively spoken, our result carries over to these frameworks.

Our non-monotonicity result accords with empirical findings of Bo and Lensink (2005). In a panel of Dutch firms, the investment–uncertainty relationship is positive at low levels of uncertainty and negative at high levels. Until now, a clear theoretical explanation for such empirical results is missing. The factors hastening investment with greater uncertainty indicated in this paper lend themselves to empirical tests.

Acknowledgment

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Appendix A. Proofs

A.1. Deterministic project life

The derivative of the investment trigger (given in (6)) with respect to σ is

$$\frac{\mathrm{d}Q^*}{\mathrm{d}\sigma} = \frac{I\beta_1}{(\beta_1-1)^2}\frac{1}{\sigma^2(\beta_1-\frac{1}{2})+\mu-\lambda\rho\sigma}\frac{1}{1-\mathrm{e}^{-(r+\lambda\rho\sigma-\mu)T}}(M-N\Delta), \tag{17}$$

where

$$\begin{aligned} M &= (\beta_1-1)(\beta_1+\tfrac{1}{2})\lambda\rho\sigma^2+(\beta_1-1)(r-\mu)\sigma+\beta_1(\mu-\lambda\rho\sigma)\lambda\rho-r\lambda\rho, \\ N &= (\beta_1-1)(\beta_1-\tfrac{1}{2})\lambda\rho\sigma^2+(\beta_1-1)(\mu-\lambda\rho\sigma)\lambda\rho, \\ \Delta &= (r+\lambda\rho\sigma-\mu)T[\mathrm{e}^{(r+\lambda\rho\sigma-\mu)T}-1]^{-1}. \end{aligned}$$

Denote the term $M - N\Delta$ by L_1. The first three fractions of (17) are always positive (recall that $\sigma^2(\beta_1 - \frac{1}{2}) + \mu - \lambda\rho\sigma = \partial L_0/\partial\beta|_{\beta=\beta_1} > 0$, as the derivative is evaluated at the higher root of the convex quadratic L_0). The sign of L_1 thus determines for the sign of the derivative. From (7) we observe that

$$(\mu - \lambda\rho\sigma)\beta_1 = -\tfrac{1}{2}\beta_1^2\sigma^2 + \tfrac{1}{2}\beta_1\sigma^2 + r,$$

which can be substituted twice into M and N to obtain

$$M = \tfrac{1}{2}(\beta_1 - 1)^2\lambda\rho\sigma^2 + (\beta_1 - 1)(r + \lambda\rho\sigma - \mu)\sigma \tag{18}$$

and

$$N = \tfrac{1}{2}(\beta_1 - 1)^2\lambda\rho\sigma^2 + (r + \lambda\rho\sigma - \mu)\lambda\rho. \tag{19}$$

Proof of Proposition 1. First, suppose that $\lambda\rho > 0$. Combining $T \to \infty$ with (17) and (18), we obtain that

$$\begin{aligned}\frac{\mathrm{d}Q^*}{\mathrm{d}\sigma} &= \frac{I\beta_1}{(\beta_1 - 1)^2}\frac{1}{\sigma^2(\beta_1 - \frac{1}{2}) + \mu - \lambda\rho\sigma}\left[(\beta_1 - 1)\sigma\left(r + \frac{1}{2}(\beta_1 + 1)\lambda\rho\sigma - \mu\right)\right]\\ &> \frac{I\beta_1}{(\beta_1 - 1)^2}\frac{1}{\sigma^2(\beta_1 - \frac{1}{2}) + \mu - \lambda\rho\sigma}[(\beta_1 - 1)\sigma(r + \lambda\rho\sigma - \mu)]\\ &\geqslant 0,\end{aligned}$$

where the first inequality stems from the observation that $\frac{1}{2}(\beta_1 + 1) > 1$ and the second from the assumption that $r + \lambda\rho\sigma - \mu = \delta > 0$.

The two other possibilities $\lambda\rho = 0$ and $\lambda\rho < 0$ are covered by the proofs of Propositions 2 and 3, respectively. □

Proof of Proposition 2. Within our model we can impose absence of the impact of systematic risk by setting $\rho = 0$. The derivative of the investment trigger (given in Eq. (6)) with respect to σ is

$$\frac{\mathrm{d}Q^*}{\mathrm{d}\sigma} = \frac{I\beta_1}{(\beta_1 - 1)^2}\frac{1}{\sigma^2(\beta_1 - \frac{1}{2}) + \mu}\frac{1}{1 - \mathrm{e}^{-(r-\mu)T}}(\beta_1 - 1)\sigma(r - \mu).$$

The resulting expression is always positive if $r > \mu$, which holds by the assumption that $\delta > 0$. □

Proof of Proposition 3. Suppose that $\lambda\rho < 0$. Then the assumption that $\delta > 0$ holds if and only if $\sigma \in [0, \bar{\sigma})$, where $\bar{\sigma} = \frac{\mu - r}{\lambda\rho}$. We have that, denoting $\delta(\cdot)$ and $\beta_1(\cdot)$ as functions of σ, $\delta(\bar{\sigma}) = 0$ and $\beta_1(\bar{\sigma}) = 1$. So $[0, \bar{\sigma})$ is the relevant domain for σ in this case. Next, we claim that

$$\tfrac{1}{2}(\beta_1 + 1)\sigma < \bar{\sigma} \quad \text{for all } \sigma \in [0, \bar{\sigma}). \tag{20}$$

To verify, note that

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{1}{2}(\beta_1+1)\sigma = \left[\sigma^2\left(\beta_1-\frac{1}{2}\right)+\mu-\lambda\rho\sigma\right]^{-1} \times \left[\frac{1}{2}(3\beta_1-1)\sigma^2+(\beta_1-1)\mu-\lambda\rho\sigma\right]>0 \quad \text{and} \quad \frac{1}{2}(\beta_1(\bar{\sigma})+1)\bar{\sigma}=\bar{\sigma}.$$

So, for positive σ less than $\bar{\sigma}$, the inequality (20) is true.

Now, $\lambda\rho<0$ implies that $N<0$. Combining (20) and (18) we have that

$$\begin{aligned} M &= (\beta_1-1)\sigma\left[r+\tfrac{1}{2}(\beta_1+1)\lambda\rho\sigma-\mu\right] \\ &> (\beta_1-1)\sigma(r+\lambda\rho\bar{\sigma}-\mu)=(\beta_1-1)\sigma\delta(\bar{\sigma})=0. \end{aligned}$$

Since $M>0$, $N<0$ and $1\geqslant\Delta>0$, the derivative (17) is also positive and the proposition is proved. □

Proof of Proposition 4. Suppose that T is finite and $\lambda\rho>0$. We want to show that L_1 is negative for low $\sigma\geqslant0$ and becomes positive when σ increases. First, it is useful to observe the simple fact that $1\geqslant\Delta>0$ and

$$\frac{\mathrm{d}\Delta}{\mathrm{d}\sigma}<0. \tag{21}$$

It can also be verified that

$$L_1\leqslant0 \Rightarrow (\beta_1-1)\sigma-\lambda\rho<0 \quad\Longleftrightarrow\quad \frac{\mathrm{d}\beta_1}{\mathrm{d}\sigma}>0. \tag{22}$$

Then note that at $\sigma=0$, $L_1=-(r-\mu)\lambda\rho\Delta<0$. So $\frac{\mathrm{d}Q^*}{\mathrm{d}\sigma}$ is also negative at $\sigma=0$. As σ increases, Δ converges to zero and L_1 becomes positive. We show now that L_1 changes its sign from negative to positive only once with increasing σ. If $L_1=0$, then $\Delta=\frac{M}{N}$ and

$$\begin{aligned} \frac{\mathrm{d}L_1}{\mathrm{d}\sigma} &= \frac{\mathrm{d}M}{\mathrm{d}\sigma}-\frac{\mathrm{d}N}{\mathrm{d}\sigma}\Delta-N\frac{\mathrm{d}\Delta}{\mathrm{d}\sigma}>\frac{\mathrm{d}M}{\mathrm{d}\sigma}-\frac{\mathrm{d}N}{\mathrm{d}\sigma}\Delta=\frac{1}{N}\left(\frac{\mathrm{d}M}{\mathrm{d}\sigma}N-\frac{\mathrm{d}N}{\mathrm{d}\sigma}M\right) \\ &= \frac{\delta\lambda\rho}{N}\left[\frac{\mathrm{d}\beta_1}{\mathrm{d}\sigma}\sigma+\beta_1-1\right]\{(\beta_1-1)[\lambda\rho-(\beta_1-1)\sigma]\sigma+\delta\} \\ &> 0, \end{aligned} \tag{23}$$

The inequalities follow from (21) and (22). So L_1 increases in σ at the point at which $L_1=0$. Now, continuity of L_1 implies that it changes its sign only once from negative to positive at some $\sigma^*>0$. Hence the first part of the proposition is proved.

To verify that the σ-interval where the negative effect occurs is larger the shorter is the project life, we consider

$$\frac{\mathrm{d}\sigma^*}{\mathrm{d}T}=-\left.\frac{\frac{\partial L_1}{\partial T}}{\frac{\partial L_1}{\partial\sigma}}\right|_{\sigma=\sigma^*}=\left.\frac{N\frac{\mathrm{d}\Delta}{\mathrm{d}T}}{\frac{\partial L_1}{\partial\sigma}}\right|_{\sigma=\sigma^*}<0.$$

The inequality follows from the fact that $\frac{\mathrm{d}\Delta}{\mathrm{d}T}<0$ and (23). □

Proof of Proposition 5. The sum of the two option effects is

$$\frac{\partial Q^*}{\partial \beta_1}\frac{\partial \beta_1}{\partial \sigma}+\frac{\partial Q^*}{\partial \beta_1}\frac{\partial \beta_1}{\partial \delta}\frac{\partial \delta}{\partial \sigma}=\frac{I\beta_1}{(\beta_1-1)^2}\frac{\delta(\sigma)}{1-\mathrm{e}^{-\delta(\sigma)T}}\frac{(\beta_1-1)\sigma-\lambda\rho}{\sigma^2(\beta_1-\frac{1}{2})+\mu-\lambda\rho\sigma}. \tag{24}$$

As $\beta_1>1$ and $\sigma^2(\beta_1-\frac{1}{2})+r-\delta(\sigma)>0$, the sign of expression (24) depends on the sign of $L_2\equiv(\beta_1-1)\sigma-\lambda\rho$ in the way stated in the proposition.

It remains to be shown that there exists a unique non-negative $\hat{\sigma}$. Note that, if $\lambda\rho>0$, at $\sigma=0$ we have that $L_2=-\lambda\rho<0$ and the combined option effect is negative. To verify that the option effect changes its sign only once from negative to positive with increasing σ, we show that L_2 (being continuous in $\sigma>0$) always increases with σ if $L_2\leqslant 0$. That is,

$$\frac{\mathrm{d}L_2}{\mathrm{d}\sigma}=\frac{\lambda\rho\sigma-(\beta_1-1)\sigma^2}{\sigma^2(\beta_1-1)+\mu-\lambda\rho\sigma}+\beta_1-1\geqslant\beta_1-1>0$$

if $L_2\leqslant 0$.

The discounting effect is given by

$$\frac{\partial Q^*}{\partial \delta}\frac{\partial \delta}{\partial \sigma}=\frac{I\beta_1}{\beta_1-1}\frac{1-\mathrm{e}^{-\delta(\sigma)T}-\delta(\sigma)T\,\mathrm{e}^{-\delta(\sigma)T}}{(1-\mathrm{e}^{-\delta(\sigma)T})^2}\lambda\rho,$$

which is always positive and increasing in T. It is straightforward from derivations leading to (24) that $\frac{\partial Q^*\partial\beta_1}{\partial\beta_1\,\partial\sigma}$ and $\frac{\partial Q^*\partial\beta_1\partial\delta}{\partial\beta_1\,\partial\delta\,\partial\sigma}$ decrease in absolute terms in T. $\square$

A.2. Stochastic project life

Let $\delta(\sigma)$ be a continuous twice differentiable convenience yield function. The derivative of Q^* given in (12) with respect to σ eventually becomes

$$\frac{\mathrm{d}Q^*}{\mathrm{d}\sigma}=\frac{I\beta_1}{(\beta_1-1)^2}\frac{1}{\sigma^2(\beta_1-\frac{1}{2})+r-\delta(\sigma)}L_3, \tag{25}$$

where

$$L_3=\tfrac{1}{2}(\beta_1-1)^2\delta'(\sigma)\sigma^2+(\beta_1-1)\delta(\sigma)\sigma+[(\beta_1-1)\sigma-\delta'(\sigma)]\gamma. \tag{26}$$

The first two fractions of the right-hand side of (25) are always positive, so the sign of the derivative is determined by the sign of L_3.

Proof of Proposition 6. The proof follows from the proof of Proposition 8 below with linear $\delta(\sigma)$. $\square$

We prove Propositions 7 and 8 only for stochastic project lifetime; similar proofs can be obtained for the deterministic case.

Proof of Proposition 7. Note that if $\gamma=0$ and $\delta'(\sigma)>0$ then $L_3=\frac{1}{2}(\beta_1-1)^2\delta'(\sigma)\sigma^2+(\beta_1-1)\delta(\sigma)\sigma>0$. $\square$

Proof of Proposition 8. We want to show that for $\gamma>0$, $\delta'(\sigma)>0$ and $\delta''(\sigma)<0$, L_3 is negative for low $\sigma\geqslant 0$ and turns to positive with increasing σ. First we note that at

$\sigma = 0$, $L_3 = -\delta'(0)\gamma < 0$. Then observe that a straightforward consequence of (26) is that

$$L_3 \leqslant 0 \Rightarrow (\beta_1 - 1)\sigma - \delta'(\sigma) < 0 \quad \Longleftrightarrow \quad \frac{d\beta_1}{d\sigma} > 0. \tag{27}$$

Using this, if $L_3 \leqslant 0$, we have that

$$\begin{aligned} \frac{dL_3}{d\sigma} &= \frac{d\beta_1}{d\sigma}[(\beta_1 - 1)\delta'(\sigma)\sigma^2 + \sigma(\delta(\sigma) + \gamma)] + (\beta_1 - 1)^2\delta'(\sigma)\sigma \\ &\quad + (\beta_1 - 1)(\delta(\sigma) + \gamma + \delta'(\sigma)\sigma) + \left[\frac{1}{2}(\beta_1 - 1)^2\sigma^2 - \gamma\right]\delta''(\sigma) \\ &> \left[-\frac{1}{\delta'(\sigma)}(\beta_1 - 1)\sigma(\delta(\sigma) + \gamma)\right]\delta''(\sigma) > 0. \end{aligned}$$

So L_3 always increases in σ if $L_3 \leqslant 0$. From the continuity of L_3 now follows that L_3 changes its sign only once from negative to positive at some $\sigma^* > 0$. This proves the first part of proposition.

To verify that the σ-interval where the negative effect occurs is larger, the shorter is the project life we consider

$$\frac{d\sigma^*}{d\gamma} = -\left.\frac{\frac{\partial L_3}{\partial \gamma}}{\frac{\partial L_3}{\partial \sigma}}\right|_{\sigma=\sigma^*} = \left.\frac{\delta'(\sigma) - (\beta_1 - 1)\sigma}{\frac{\partial L_3}{\partial \sigma}}\right|_{\sigma=\sigma^*} > 0,$$

where for the inequality we employ (27) and the first part of the proof of this proposition. □

Appendix B. Supplementary data

Supplementary data associated with this article can be found in the online version at 10.1016/j.jedc.2007.10.003.

Appendix A. Supplementary data

Application 1.

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