Modeling Dynamic Effects of Promotion on Interpurchase Times

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Abstract

In this paper we put forward a duration model to analyze the dynamic effects of marketing-mix variables on interpurchase times. We extend the accelerated failure time model with an autoregressive structure. An important feature of the model is that it allows for different long-run and short-run effects of marketing-mix variables on interpurchase times. As marketing efforts usually change during the spells, we explicitly deal with time-varying covariates. Our empirical analysis of purchases of liquid laundry detergent reveals that the short-run effects of marketing-mix variables are significantly different from the long-run effects.

Key words: dynamic duration model, time-varying covariates, error-correction model, unobserved heterogeneity.

JEL codes: C41, C23

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*We thank Teun Kloek for helpful comments. The models described in this paper are implemented in Ox 3.00 (Doornik 1999). Address for correspondence: Dennis Fok, Erasmus University Rotterdam, Econometric Institute H11-2, P.O. Box 1738, NL-3000 DR Rotterdam, The Netherlands, e-mail: dfok@few.eur.nl

1 Introduction

For marketing managers it is important to understand the dynamic effects of marketing-mix variables like promotion and advertising on marketing performance measures such as sales, market shares and profitability. Particularly, it is relevant to understand the long-run effects of marketing efforts as this knowledge can for example lead to more efficient marketing strategies. Examples of recent studies that address this issue are Mela et al. (1997), Dekimpe et al. (1999), Jedidi et al. (1999) and Paap and Franses (2000) to mention just a few.

In this paper we address the issue of measuring the long-run impact of marketing-mix variables in the context of interpurchase times. The theoretical and empirical analysis of purchase-timing behavior of households has received considerable attention in recent years. Using an extension of an Erlang-2 type of model, Gupta (1988) demonstrates that promotions shorten interpurchase times. Helsen and Schmittlein (1993), Jain and Vilcassim (1991) and Vilcassim and Jain (1991) propose to analyze interpurchase times using hazard functions. The last two studies also incorporate household heterogeneity, see also for example Gönül and Srinivasan (1993) for further refinements. An important extension to modeling interpurchase times for two related product categories is given in Chintagunta and Haldar (1998). Furthermore, Chintagunta and Prasad (1998) consider modeling purchase-timing behavior and brand-choice decisions jointly.

Dynamic models for interpurchase times are however relatively scarce. This is however strange as one may expect an increase in interpurchase times after a promotion. One example of a study that explicitly incorporates dynamic structures is Allenby et al. (1999). In this paper dynamics in durations are modeled by lagged interpurchase times. In contrast to the above-mentioned studies concerning dynamics in sales and market shares, this study on interpurchase times does not explicitly consider separating long-run from short-run effects of marketing mix variables on purchase-timing behavior. In the present paper we aim to contribute to the literature by putting forward a dynamic model for interpurchase times that does allow for such an interpretation. The key feature of our model is that it allows the long-run effects to differ from short-run effects. The model extends the familiar accelerated failure-time model by including lagged interpurchase times as well as lagged covariates. Rewriting this model as an Error Correction Model [ECM] allows us to distinguish the long-run from short-run effects, see Hendry et al. (1984).

The value of marketing mix-variables, like price and promotion, are likely to change during interpurchase spells. In most marketing applications of duration models it is assumed that covariates remain constant during spells. In this paper we follow a similar approach as Gupta (1991) to allow for time-varying covariates in the hazard specification. Additionally, many studies have emphasized the relevance of household heterogeneity. We accommodate for unobserved differences across households by a latent class approach.

The outline of our paper is as follows. In Section 2, we discuss our dynamic duration model. We show how the accelerated failure-time model can be extended to allow for time-varying covariates and possibly differing long-run and short-run effects of marketing variables. We discuss in detail how one can interpret the parameters and estimate them using maximum likelihood. In Section 3, we apply our model to purchases of liquid laundry detergent. One of our main empirical findings is that the short-run effects of marketing mix variables are significantly different from the long-run effects. In Section 4, we conclude our paper with a discussion of the main results with suggestions for further research topics.

2 A Dynamic model for Interpurchase Times

In this section we put forward our dynamic model for interpurchase times, which enables a separate evaluation of long-run and short-run effects of promotion and other marketing-mix variables. In Section 2.1, we present the functional form of the hazard specification and discuss the handling of time-varying covariates. In Section 2.2, we introduce autoregressive dynamics in our model. The interpretation of the dynamic structure is discussed in Section 2.3. Finally, in Section 2.4, we consider parameter estimation.

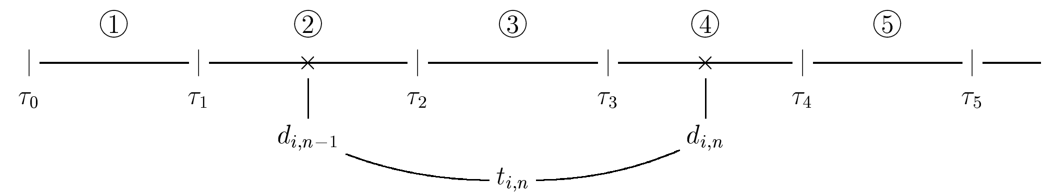

Figure 1: Graphical representation of purchase occasion $d_{i,n}$, interpurchase time $t_{i,n}$ and time indexes of changes in covariates τ_l

2.1 Hazard Specification

Assume that a household $i = 1, \ldots, I$ purchases a certain product at time $d_{i,n}$, for $n = 0, \ldots, N_i$ over a certain period of time. The N_i interpurchase times of this household are therefore defined by $t_{i,n} = d_{i,n} - d_{i,n-1}$ with $n = 1, \ldots, N_i$. To model the interpurchase times we consider a hazard specification. Denote the hazard corresponding to the n-th purchase decision of household i by

$$\lambda_{i,n}(t|x_{i,n}(t), \theta_i), \tag{1}$$

where $x_{i,n}(t)$ denotes a vector of covariates explaining the hazard of household i for the n-th purchase decision at time t and θ_i is a household-specific parameter vector. The explanatory variables are a function of time t. Note that the variable t in $x_{i,n}(t)$ gives the value of the covariates at time $d_{i,n-1} + t$.

For modeling interpurchase times it a rather unrealistic to assume that the covariates are constant during the spell, see Gupta (1991). The set of covariates will usually include marketing instruments such as price and display. These variables usually change per week or at most per day. Denote by τ_l for $l = 0, .., L$ the time indexes where there is a change in one of the covariates. For ease of exposition assume that the covariates are constant over a period of a week. Week 1 then corresponds to the time interval $[\tau_0, \tau_1]$. Denote by $K_{i,n}(t)$ the week number corresponding to t time periods after the start of the n-th spell of household i, this week starts at $\tau_{K_{i,n}(t)-1}$ and ends at $\tau_{K_{i,n}(t)}$. In Figure 1 we give a graphical representation of the purchase process. In this example we have a purchase in weeks 2 and 4, in this case we would have $K_{i,n}(0) = 2$ and $K_{i,n}(t_{i,n}) = 4$.

To derive the distribution of the interpurchase times we use the fact that the survivor function $S_{i,n}(t|x_{i,n}(t),\theta_i))$ equals $\exp(-\Lambda_{i,n}(t|x_{i,n}(t),\theta_i))$, where $\Lambda_{i,n}(t|x_{i,n}(t),\theta_i)$ is the integrated hazard function. This function is defined as

$$\Lambda_{i,n}(t|x_{i,n}(t),\theta_i) = \int_0^t \lambda_{i,n}(u|x_{i,n}(u),\theta_i)du. \tag{2}$$

Note that the integrated hazard function depends on the whole path of $x_{i,n}(u)$ for $u = 0,\ldots,t$, see Lancaster (1990) for a discussion. This integral can be decomposed by identifying intervals in which $x_{i,n}(t)$ is constant. We decompose the integral in three parts, (i) from the start of the duration to the end of the corresponding week, (ii) weeks completely contained in the duration, (iii) from the start of the final week to the end of the duration. The integrated hazard can be decomposed as,

$$\Lambda_{i,n}(t|x_{i,n}(t),\theta_i) = \int_0^{\tau_{K_{i,n}(0)}-d_{i,n-1}} \lambda_{i,n}(u|x_{i,n}(u),\theta_i)du$$
$$+ \sum_{k=K_{i,n}(0)}^{K_{i,n}(t)-2} \int_{\tau_k-d_{i,n-1}}^{\tau_{k+1}-d_{i,n-1}} \lambda_{i,n}(u|x_{i,n}(u),\theta_i)du + \int_{\tau_{K(t)-1}-d_{i,n-1}}^{t} \lambda_{i,n}(u|x_{i,n}(u),\theta_i)du, \tag{3}$$

see Gupta (1991) for a similar approach.

As the computation of the integrated hazard function is computational intensive, it is convenient to have a closed form expression for the individual elements of (3). Therefore, in this paper we specify the hazard as an accelerated failure time hazard with a log-logistic baseline hazard. This hazard specification allows for an analytical expression of the integrated baseline hazard and allows for a non-monotonic hazard function. The advantage of using an accelerated failure time specification is that it can be written in a linear specification, see Kalbfleisch and Prentice (1980), Kiefer (1988) and Ridder (1990). This facilitates the inclusion and interpretation of an autoregressive dynamic structure in the model, see also Engle and Russell (1998) who consider a similar approach in modeling financial transaction data. The hazard now becomes

$$\lambda_{i,n}(t|x_{i,n}(t),\theta_i) = \exp(-x_{i,n}(t)'\beta_i)\lambda_0(t\exp(-x_{i,n}(t)'\beta_i)|\delta_i), \tag{4}$$

where $\lambda_0(t|\delta)$ denotes the baseline hazard, which in case of the log-logistic distribution is defined as

$$\lambda_0(t|\delta) = \frac{\delta t^{\delta-1}}{1+t^{\delta}}. \tag{5}$$

and where $\theta_i = (\beta_i, \delta_i)$. For completeness, the hazard function becomes

$$\begin{aligned}\lambda_{i,n}(t|x_{i,n}(t), \theta_i) &= \exp(-x_{i,n}(t)'\beta_i)\lambda_0(t\exp(-x_{i,n}(t)'\beta_i)) \\ &= \frac{\delta_i t^{\delta_i - 1}\exp(-x_{i,n}(t)'\beta_i)^{\delta_i}}{1 + t^{\delta_i}\exp(-x_{i,n}(t)'\beta_i)^{\delta_i}}.\end{aligned} \tag{6}$$

If the covariates for the n-th spell of household i are constant over the interval $[a, b)$, the integrated baseline hazard equals

$$\begin{aligned}&\int_a^b \lambda_{i,n}(u|x_{i,n}(a), \theta_i)du = \\ &\qquad \log[1 + b^{\delta_i}\exp(-x_{i,n}(a)'\beta_i)^{\delta_i}] - \log[1 + a^{\delta_i}\exp(-x_{i,n}(a)'\beta_i)^{\delta_i}].\end{aligned} \tag{7}$$

This result can be used to compute (3). The density function for observation $t_{i,n}$ can be expressed in terms of the hazard function and integrated hazard function

$$f_{i,n}(t_{i,n}|x_{i,n}(t), \theta_i) = \lambda_{i,n}(t_{i,n}|x_{i,n}(t), \theta_i)S_{i,n}(t_{i,n}|x_{i,n}(t), \theta_i), \tag{8}$$

where $S_{i,n}(t_{i,n}|x_{i,n}(t), \theta_i) = \exp(-\Lambda_{i,n}(t_{i,n}|x_{i,n}(t), \theta_i))$ denotes the survival function.

2.2 Dynamics

The model discussed in the previous section is static, in the sense that interpurchase times are only explained by current explanatory variables. It is however likely that the interpurchase times of households are correlated over time. For example, promotional activities may not only have an effect on current but also on future interpurchase times.

A very flexible specification of these dynamical patterns is obtained by adding lagged interpurchase times and the value of the marketing instruments at the last purchase to the vector of covariates. To explain the effect of lagged interpurchase times and lagged covariates on current interpurchase time, we assume for the moment that the covariates are constant during the spell, that is, $x_{i,n}(t) = x_{i,n}$. In this case, the survival function simplifies to

$$S_{i,n}(t|x_{i,n}, \theta_i) = \frac{1}{1 + [\exp(-x_{i,n}'\beta_i)t]^{\delta_i}}. \tag{9}$$

Now consider the distribution of $t_{i,n}^* = \delta_i(\ln t_{i,n} - x'_{i,n}\beta_i)$

$$
\begin{aligned}
\Pr[t_{i,n}^* < E] &= \Pr[\delta_i(\ln t_{i,n} - x'_{i,n}\beta_i) < E] = \Pr[t_{i,n} < \exp(x'_{i,n}\beta_i + 1/\delta_i E)] \\
&= 1 - S(\exp(x'_{i,n}\beta_i + 1/\delta_i E)) = 1 - \frac{1}{1+\exp(1/\delta_i E)_i^\delta} \\
&= 1 - \frac{1}{1+\exp(E)}.
\end{aligned} \tag{10}
$$

Hence, in the case of constant regressors, $t_{i,n}^*$ has a logistic density. As its density does not depend on covariates and model parameters we can linearize the duration model as follows

$$
\ln t_{i,n} = x'_{i,n}\beta_i + \sigma_i u_{i,n}, \tag{11}
$$

where $\sigma_i = 1/\delta_i$ and where $u_{i,n}$ is logistic distributed such that $\mathrm{E}[u_{i,n}] = 0$.

The model in (11) is static, in the sense that interpurchase times are only explained by current explanatory variables. To model dependence between the current interpurchase time and the previous interpurchase time, we may allow for an autoregressive structure of the error term, that is,

$$
u_{i,n} = \rho_i u_{i,n-1} + \eta_{i,n}, \tag{12}
$$

which transforms (11) into

$$
(\ln t_{i,n} - x'_{i,n}\beta_i) = \rho_i(\ln t_{i,n-1} - x'_{i,n-1}\beta_i) + \sigma_i\eta_{i,n}. \tag{13}
$$

The autoregressive parameter ρ_i is equal to the correlation $\mathrm{E}[(\ln t_{i,n} - x'_{i,n}\beta_i)(\ln t_{i,n-1} - x'_{i,n-1}\beta_i)]$, and $\eta_{i,n}$ is again an unobserved error term for which we take the same distributional assumptions as for $u_{i,n}$. To exclude explosive behavior of the interpurchase times we impose that $|\rho_i| < 1$ for all i. To highlight the dynamic properties of (13), it can be rewritten in the so-called error correction format, that is,

$$
\Delta \ln t_{i,n} = \Delta x'_{i,n}\beta_i + (\rho_i - 1)(\ln t_{i,n-1} - x'_{i,n-1}\beta_i) + \sigma_i\eta_{i,n}, \tag{14}
$$

where Δ is the first difference operator defined as $\Delta z_{i,n} = z_{i,n} - z_{i,n-1}$, where $z_{i,n}$ can be $\ln t_{i,n}$ or $x_{i,n}$. The term $\Delta x'_{i,n}\beta_i$ concerns the short-run effects of a change in $x_{i,n}$ on the interpurchase time, while the term $-x'_{i,n-1}\beta_i$ in the so-called error correction part concerns the long-run effects. As these two terms involve the same parameter β_i, from (14) it is clear that model (13) assumes that short-run effects of changes in $x_{i,n}$ are the same as the long-run effects. Therefore, we will refer to model (13) as the common factor representation, thereby following the usual time series terminology, see Hendry et al. (1984).

In many situations, it is however quite possible that for example promotional activities have long-run effects which may differ in size or even in sign from short-run effects. To allow for different short- and long-run effects in our model for interpurchase times, we generalize (14) by replacing $\Delta x'_{i,n}\beta_i$ by $\Delta x'_{i,n}\alpha_i$ resulting in the following error correction duration model

$$\Delta \ln t_{i,n} = \Delta x'_{i,n}\alpha_i + (\rho_i - 1)(\ln t_{i,n-1} - x'_{i,n-1}\beta_i) + \sigma_i \eta_{i,n}. \tag{15}$$

Notice that we cannot estimate different short- and long-run effects of variables that do not change during the time period considered, like for example household size, as then $\Delta x_{i,n}$ will be zero and α_i is not identified.

Another option to allow for correlation in interpurchase times is to add $\ln t_{i,n-1}$ as an explanatory variable to (11). This, however, leads to a dynamic specification which is difficult to interpret as the parameters do not correspond directly with short- and long-run effects. Furthermore, it is easy to show that if one would include both $x_{i,n-1}$ and $\ln t_{i,n-1}$ as explanatory variables in (11), the resulting model has the same representation as (15).

For notational convenience, we write (15) as

$$\begin{aligned} \ln t_{i,n} &= x'_{i,n}\alpha_i + \rho_i \ln t_{i,n-1} + x'_{i,n-1}(-\alpha_i - (\rho_i - 1)\beta_i) + \sigma_i \eta_{i,n} \\ &= w'_{i,n}\gamma_i + \sigma_i \eta_{i,n}, \end{aligned} \tag{16}$$

where $w_{i,n}$ contains $(x_{i,n}, \ln t_{i,n-1}, x_{i,n-1})$ and $\gamma_i = (\alpha_i, \rho_i, -\alpha_i - (\rho_i - 1)\beta_i)$.

The hazard function in the case of the ECM specification can easily be obtained from (6) by replacing $x_{i,n}(t)'\beta_i$ by $w_{i,n}(t)'\gamma_i$. Indeed, the results for the common factor model (14) and the static model (11) can be obtained by restricting $\alpha_i = \beta_i$ and by setting $\alpha_i = \beta_i$ and $\rho_i = 0$, respectively. The hazard corresponding with this dynamic duration model has to be defined as a conditional hazard given the previous interpurchase time.

This conditional hazard function for $t_{i,n}$ given $t_{i,n-1}$ reads as

$$\lambda_{i,n}(t|w_{i,n}(t),\theta_i) = \frac{\delta_i t^{\delta_i - 1}\exp(-w_{i,n}(t)'\gamma_i)^{\delta_i}}{1 + t^{\delta_i}\exp(-w_{i,n}(t)'\gamma_i)^{\delta_i}} \tag{17}$$

with $\theta_i = (\gamma_i, \delta_i)$. The density function of the timing of the n-th purchase occasion of household i given $t_{i,n-1}$ is therefore

$$f_{i,n}(t|w_{i,n}(t),\theta_i) = \lambda_{i,n}(t|w_{i,n}(t),\theta_i)\exp(-\Lambda_{i,n}(t|w_{i,n}(t),\theta_i)), \tag{18}$$

where $\Lambda_{i,n}(t|w_{i,n}(t),\theta_i)$ is the integrated hazard function.

2.3 Interpretation of Parameters

In this section, we analyze the dynamic effects of the explanatory variables on interpurchase times. The short-run effect of a marketing instrument is defined as the instantaneous effect of a (permanent) change on the interpurchase time. The long-run effect measures the effect of a permanent change of a marketing instrument at time t' on the interpurchase times at t as $t \to \infty$. We focus on the error correction duration model (15) as this model nests the common factor representation (13) ($\alpha_i = \beta_i$) and the static model (11) ($\alpha_i = \beta_i$ and $\rho_i = 0$).

First, we consider the derivative of $\ln t_{i,n}$ with respect to $x_{i,n}$, that is,

$$\frac{\partial \ln t_{i,n}}{\partial x_{i,n}} = \alpha_i \tag{19}$$

Hence, an ε change in $x_{i,n}$, for example due to a price reduction or a promotional activity, leads to $\alpha_i\varepsilon$ change in the log current interpurchase time. Note that if $x_{i,n}$ is for example the log of a variable, we can interpret α_i as an elasticity.

To analyze the effects of changes in the explanatory variables on future log interpurchase times, we can follow a similar procedure. The partial derivative of $\ln t_{i,n+1}$ with respect to $x_{i,n}$ is given by

$$\frac{\partial \ln t_{i,n+1}}{\partial x_{i,n}} = -\alpha_i - (\rho_i - 1)\beta_i + \rho_i\frac{\partial \ln t_{i,n}}{\partial x_{i,n}} = (\rho_i - 1)(\alpha_i - \beta_i). \tag{20}$$

An ε change in $x_{i,n}$ leads to a change of $\varepsilon(\rho_i - 1)(\alpha_i - \beta_i)$ in $\ln t_{i,n+1}$. The derivative is zero if $\alpha_i = \beta_i$. Hence, the common factor specification (14) (and of course the static model in (11)) imposes that changes in $x_{i,n}$ have no effect on the next interpurchase time. The same is true for the subsequent interpurchase time as

$$\frac{\partial \ln t_{i,n+2}}{\partial x_{i,n}} = \rho_i \frac{\partial \ln t_{i,n+1}}{\partial x_{i,n}} = \rho_i(\rho_i - 1)(\alpha_i - \beta_i). \tag{21}$$

To derive the partial derivative of $\ln t_{i,n+k}$ with respect to $x_{i,n}$ we note that for $r > 2$ $\partial \ln t_{i,n+r}/\partial x_{i,n} = \rho_i \partial \ln t_{i,n+r-1}/\partial x_{i,n}$ and hence that

$$\frac{\partial \ln t_{i,n+k}}{\partial x_{i,n}} = \rho_i^{(k-1)}(\rho_i - 1)(\alpha_i - \beta_i). \tag{22}$$

If $|\rho_i| < 1$ the effect of a change in $x_{i,n}$ on future interpurchase times will decline exponentially and eventually it becomes zero.

From the above exercise it can already be understood that permanent changes in interpurchase times can only be obtained when $x_{i,n}$ changes permanently. For example, our model implies that only a permanent lower price can generate a permanent reduction in interpurchase times. To derive the long-run effects of a permanent change in $x_{i,n}$, we apply repeated backward substitution to (15) and obtain

$$\begin{aligned}\ln t_{i,n} &= \rho_i \ln t_{i,n-1} + \Delta x'_{i,n}\alpha_i - (\rho_i - 1)x'_{i,n-1}\beta_i + \sigma_i \eta_{i,n} \\ &= \rho_i^2 \ln t_{i,n-2} + \Delta x'_{i,n}\alpha_i + \rho_i \Delta x'_{i,n-1}\alpha_i \\ &\qquad - (\rho_i - 1)x'_{i,n-1}\beta_i - \rho_i(\rho_i - 1)x'_{i,n-2}\beta_i + \sigma_i\eta_{i,n} + \rho_i\sigma_i\eta_{i,n-1} \\ &= \rho_i^n \ln t_{i,0} + \sum_{j=0}^{n-1} \rho_i^j (\Delta x'_{i,n-j}\alpha_i - (\rho_i - 1)x'_{i,n-j-1}\beta_i + \sigma_i\eta_{i,n-j}),\end{aligned} \tag{23}$$

where $t_{i,0}$ denotes the pre-sample starting value of $t_{i,n}$. As $|\rho_i| < 1$, $\rho_i^n \to 0$ for large n and the influence of $\ln t_{i,0}$ can be neglected. If we further assume that $x_{i,n}$ is fixed over the purchase occasions, that is $x_i = x_{i,n} = x_{i,n-j}$, $j = 1, \ldots, \infty$, then for $n \to \infty$, (23) becomes equal to

$$\ln t_{i,n} = \sum_{j=0}^{\infty} \rho_i^j (-(\rho_i - 1)x'_i\beta_i + \sigma_i\eta_{i,n-j}) = x'_i\beta_i + \sum_{j=0}^{\infty} \rho_i^j \sigma_i \eta_{i,n-j}. \tag{24}$$

Hence, as $\mathrm{E}[\eta_{i,n-j}] = 0$ for all j, the long-run expectation of $\ln t_{i,n}$ given x_i is

$$\mathrm{E}[\ln t_{i,n}|x_i] = x'_i\beta_i. \tag{25}$$

If follows from (25) that the long-run effect of a permanent change in x_i on the log interpurchase time is β_i. In sum, our error correction model for interpurchase times has short-run effects α_i and long-run effects β_i. In the common factor model these effects both equal β_i and in the static model there are no dynamic effects.

2.4 Parameter Estimation

Differences in interpurchase times across households may partly be captured by including household specific explanatory variables in the model. Furthermore, it is also not unlikely that households may react differently to promotional activities. Therefore, we have allowed for household-specific α_i and β_i parameters. Using similar arguments as for brand choice, neglecting this household heterogeneity may lead to an overestimate of the persistence (in our case ρ_i) in interpurchase times. See for example Keane (1997) for a discussion of the effects of neglecting household heterogeneity on state dependence in brand choice.

Estimation of these household-specific parameters may however be difficult if we do not have enough observations for each household. To circumvent this problem, one usually assumes that α_i and β_i are draws from a certain population distribution. This approach is followed in the brand choice models in for example Kamakura and Russell (1989), Chintagunta et al. (1991) and Gönül and Srinivasan (1993) among others.

A convenient choice is to assume that α_i and β_i are draws from a finite mixture distribution which approximates the household heterogeneity distribution, see Jain et al. (1994) and Allenby and Rossi (1999) among others. The density function for household i then becomes

$$g_i(t_{i,1}, \ldots, t_{i,N_i}|\theta) = \sum_{m=1}^{M} p_m h_i(t_{i,1}, \ldots, t_{i,N_i}|\theta_m), \tag{26}$$

where M denotes the number of mixture components with $0 < p_m < 1$, $m = 1, \ldots, M$ and $\sum_{m=1}^{M} p_m = 1$, and where θ collects the parameters and $h(t_{i,1}, \ldots, t_{i,N_i}|\theta_m)$ is the density function conditioned on segment m, defined as

$$h_i(t_{i,1},\ldots,t_{i,N_i}|\theta_m) = \\ f_{i,1}(t_{i,1}|x_{i,1},\theta_m) S_{i,N_i}(t_{i,N_i}|w_{i,N_i}(t),\theta_m) \prod_{n=2}^{N_i-1} f_{i,n}(t_{i,n}|w_{i,n}(t),\theta_m), \quad (27)$$

where the density function $f_{i,n}(t_{i,n}|w_{i,n}(t),\theta_m)$ is given in (18). The second term, involving the survivor function, is included for the last observation of household i when it is censored from the right, see for example Kiefer (1988) for a discussion. If there is no censoring, one can simply remove this term and replace the upper limit of the sum by N_i. The density for the first observation is denoted by $f_{i,1}(t_{i,1}|x_{i,1},\theta_m)$. For the first interpurchase time we do not observe the lagged interpurchase time. We choose to model the initial observation using the long-run relation between interpurchase times and the marketing instruments in (25). To be more specific the initial observation is modeled using

$$\ln t_{i,1} = \mu_{0,i} + x'_{i,1}\beta_i + \tilde{\sigma}_i \eta_{i,1}, \quad (28)$$

where $\eta_{i,1}$ has a logistic distribution. Note that we allow for a different intercept and scale parameter for the initial observation.

The parameters of duration models generally can be estimated using maximum likelihood [ML]. The log likelihood function is given by

$$\ell(\theta) = \sum_{i=1}^{I} \ln(g_i(t_{i,1},\ldots,t_{i,N_i}|\theta)) \quad (29)$$

where $g_i(t_{i,1},\ldots,t_{i,N_i}|\theta)$ is defined in (26). This log likelihood function can be maximized using standard numerical optimization algorithms. In case of household heterogeneity one may opt for the EM-algorithm of Dempster et al. (1977). The resulting maximum likelihood estimator denoted by $\hat{\theta}$ is normally distributed with mean θ and the information matrix as covariance matrix. To compute this covariance matrix, we take the outer product of gradients.

Parameter estimates for the static duration model (11) and the common factor duration model (14) can be obtained in a similar way. As both models are nested in the error correction model (15), we can use standard likelihood ratio tests to compare the three models. For instance, under the parameter restriction $\alpha_m = \beta_m$ for $m = 1, \ldots, M$ the error correction duration model (15) simplifies to the common factor model (14). To compare both models, we can perform a likelihood ratio test for the hypothesis $\alpha_m = \beta_m$. The corresponding likelihood ratio test statistic, is asymptotically $\chi^2(J)$ distributed under the null hypothesis, where J denotes the number of parameter restrictions.

It should be stressed that the likelihood ratio test procedure to compare two model specifications, is only valid if the two models under consideration are nested, in this case they should have the same number of mixture components M to describe household heterogeneity. If the number of mixture components is different in the two model specifications, the test includes a test for the number of mixture components M. Likelihood ratio tests for the number of mixture components M are not asymptotically χ^2 distributed. To illustrate this, consider a common factor model with two mixture components ($M = 2$). Under the restriction $\beta_1 = \beta_2$ the mixing proportion p_1 is not identified and the likelihood ratio test statistic for $\beta_1 = \beta_2$ is not asymptotically χ^2 distributed under the null hypothesis. This phenomenon is known as the Davies (1977) problem. We will abstract from a further analysis of this issue here, and in our empirical work we will use the BIC to determine the value of M following the approach of Jain et al. (1994).

3 Application

In this section we illustrate the dynamic duration models on a scanner panel on purchases of liquid detergent. In Section 3.1, we discuss the data set. In Section 3.2, we consider the maximum likelihood estimates of various duration models and we examine the presence of dynamic effects in interpurchase times. In Section 3.3, we use the estimation results to analyze the short-run effects of promotions on interpurchase times.

3.1 The Data

The data we use are A.C. Nielsen household scanner panel data on purchases of liquid detergent in Sioux Falls, South Dakota. A subset of these data are analyzed in Chintagunta and Prasad (1998) using a Dynamic McFadden Model. In fast moving consumer goods markets marketing efforts tend to be constant during a week, where the week is defined from Wednesday to Tuesday. We therefore aggregate our data to a weekly level. In case a household makes several purchases within a week, we use the volume bought weighted purchase date.

The sample contains 786 households with, aggregated to a weekly level, 8239 purchases from August 1986 to July 1988. Of the 35 different brands of detergent bought in this time frame, we only have complete information on marketing efforts of 13 brands. These 13 brands account for 86% of the total market. Households are selected to purchase only from these 13 brands. For each purchase occasion, we know the time since the last purchase of detergent and the volume of detergent bought. Furthermore, for each week we know the shelf price (dollars/32oz.) of these brands and which brands are featured or displayed. All marketing instruments are aggregated over available stores in the region and per week. The display and feature variables therefore represent the percentage of stores featuring the brand, or having the brand on display. Finally, we know the household size, household income and the volume purchased on the previous purchase occasion.

The interpurchase times not only depend on marketing efforts at the purchase occasion, they are also influenced by all efforts in the weeks in between purchases. For these weeks we do not know which brand will be bought. Following Gupta (1991) we use household specific brand shares to obtain marketing mix-variables at the category level. These category level variables are also used for the weeks at which a purchase is made. Note that by using household specific shares we exclude the case that the prices of brand A are used to explain interpurchase times of a specific household when in fact this household is always buys brand B.

3.2 Estimation Results

We focus on the dynamic effects of promotions on interpurchase times for the detergent product category. We do not intend to model brand choice and purchase quantity. To analyze interpurchase times, we consider the static duration model (11), the common factor duration model (13), and the error correction duration model (15). As explanatory variables we use household size, household income and the volume purchased on the previous purchase occasion (divided by 32 oz.). The latter variable is used as a proxy for "regular" and "fill-in" trips and to take into account the effects of household inventory behavior on purchase timing, respectively, see also Chintagunta and Prasad (1998). Furthermore, we include the regular price and price cuts in dollars per 32 ounce. Finally three variables are used to indicate whether brands were on display or were featured on a line or major advertisement, respectively.

We first consider models without household heterogeneity ($M = 1$). Table 1 displays the maximum likelihood estimates of the parameters of the static duration model (11), the common factor duration model (14) and the error correction duration model (15). This table also contains the maximum likelihood values and the values of the familiar Akaike [AIC] and Bayesian Information Criteria [BIC]. The second column of this table displays the parameter estimates for the static model. It shows that, as expected, household size has a significantly negative effect on interpurchase times. Household income also negatively influences interpurchase times. Wealthier and larger households tend to buy liquid detergent more often. Not surprisingly, the volume purchased on the previous purchase occasion has a significant positive effect. Display, features and price cuts naturally negatively influence interpurchase times. Line advertisements do not have a significant impact. Regular price has a positive effect. A high regular price seems to reduce the propensity to buy and thereby increases interpurchase times.

The third column shows the parameter estimates for the common factor duration model (14). The autoregressive coefficient ρ is equal to 0.35 and hence there is significant correlation in interpurchase times. The signs of the parameters are the same as those for model (11). However, the magnitude of the effects of the marketing mix variables change. The effects of price and display increase while the effects of line and major advertisements decrease. The estimated standard errors of these parameters, however, are roughly the same as in the static model. There is more evidence for significant effects of price and display on log interpurchase times. In the common factor model no advertisement variable remains significant. The difference in maximum likelihood values between the static and the common factor model indicates that the restriction $\rho = 0$ is overwhelmingly rejected at the 5% level of significance (the 95% percentile of the $\chi^2(1)$ distribution is 3.84).

The final column of Table 1 displays the parameter estimates of the error correction duration model (15) which allows for different short-run and long-run effects of explanatory variables. As household size and income are constant over the time period considered, we cannot estimate a different short-run and long-run effect of this variable. The likelihood ratio statistic to compare the error correction model with the common factor model equals $-2 \times (21\ 707.65 - 21\ 720.67) = 26.04$, which is significant at the 5% level of significance (the 95% percentile of the $\chi^2(6)$ distribution equals 12.59). This suggests that there are different short-run and long-run effects of the explanatory variables. If we consider the estimated standard errors of the parameters, we see that the short-run effect of display, denoted by α, is significant, while the long-run effect, denoted by β, is not. Major advertisements only have long term effects. Regular price, price cuts and the volume previously bought have short-term as well as long-term effects. The estimate of the ρ parameter does not differ much from that of the common factor specification.

In the models discussed so far, we assumed that there was no household heterogeneity. Table 2 shows the maximum likelihood estimates of the three duration models, where we now do allow for such heterogeneity. Due to identification issues discussed before, it is not possible to test for the number of mixture components using standard χ^2-distributed likelihood ratio test statistics. To determine the number of mixture components, we follow the strategy by Jain et al. (1994) and add extra mixture components until the BIC stops decreasing.

For both dynamic models and the static model the information criterions indicate that three mixture components are sufficient. Table 2 shows the estimated average effect of the explanatory variables on log interpurchase times, that is $\hat{\theta} = \sum_{m=1}^{3} \hat{p}_m \hat{\theta}_m$, where θ again denotes a general parameter. In general, for all three model specifications the signs of the effects are the same as for the models without household heterogeneity in Table 1. Only for line advertisements do we observe a change in sign, but this variable is not significant in all models.

The difference in the maximum likelihood values of the static and the dynamic model shows again that the static specification ($\rho = 0$) is clearly rejected at the 5% level of significance. Furthermore, the estimates of the effects are about the same as for the model without household heterogeneity.

The autoregressive coefficient is estimated to be smaller than that for the specification without heterogeneity. Hence, neglecting household heterogeneity leads to a over-estimation of the persistence in log interpurchase times. In both dynamic models the autoregressive coefficient remains significantly different from zero. Furthermore note that the reported estimate concerns the population average. Unreported results show that there exists a segment of households (9% of the population) with a considerably stronger persistence.

The final column of Table 2 displays the estimation results for the error correction specification with different short-run and long-run effects. The likelihood ratio test statistic to compare the common factor model with this model equals $-2 \times (21\,004.38 - 21\,021.85) = 34.94$, which is significant when compared to the 95% percentile of the $\chi^2(3 \times 6)$ distribution. This indicates that the short-run effects of the explanatory variables are again significantly different from the long-run effects. If we consider the estimated standard errors, we notice that of the marketing instruments only the regular price and price cuts have a significant long-run and short-run effect. Display only has a significant short-run effect while on avererage advertisements do not significantly influence interpurchase times. Note that we have only discussed the average effects. For example in the third segment (34% of the population) there is a significant short-run effect of major advertisements.

3.3 Short-Run Effects of Promotions

The estimated duration models can be used to analyze the short-run effects of promotional activities on interpurchase times. In this section we analyze the effects of a price reduction, a display, and a feature on future interpurchase times. The results are based on the parameter estimates of the error correction model (15) with household heterogeneity as displayed in the final column of Table 2.

Figure 2 shows the effect of a unit price cut at $n = 1$ on current and future log interpurchase times. The solid line in the figure displays the partial derivatives of log interpurchase times to the price cut for the current and 9 future interpurchase occasions, as described in Section 2.3. We see that the price reduction has a strong negative effect on the current interpurchase time. The effect on future purchase however is almost zero. This can also be seen in Table 2 as the short-run effect almost equals the long-run effect. The two dotted lines display the cumulative and average effect, that is, the sum of the past and current derivatives and the sum of the past and current derivatives divided by the number of purchases, respectively. After 10 purchases the cumulative effect is negative, which means that the total effect of the price reduction is negative.

Figure 3 shows the effect of a display for current and future log interpurchase times. A display at $n = 1$ has a negative effect on the current log interpurchase, but a strong positive effect on the next log interpurchase time. After a display households tend to delay their next purchases for liquid detergent. After 4 purchases, the effect of the display is almost zero. The cumulative effect after 10 purchases is negative, which means that there is a total negative effect of a display.

Finally, Figure 4 shows the effect of a major advertisement on current and future log interpurchase times. A feature has a negative effect on the current and next log interpurchase times. Again, after 4 purchases the effect of the feature is almost zero. Contrary to the effects of display, the direct effect of a major advertisement is not compensated by an effect of an opposite sign on future log interpurchase times. The cumulative effect of such an advertisement is therefore large. After 10 periods the average effect is still relatively large.

4 Conclusion

In this paper we proposed a dynamic model for interpurchase times, in which we can disentangle short-run from long-run effects of marketing variables. We discussed representation, interpretation and estimation issues. We illustrated our model for purchases on liquid detergents and we found that the short-run effects of marketing-mix variables are significantly different from the long-run effects. Additionally, we showed that our model can be used to evaluate marketing strategies.

There are at least two potentially fruitful areas for further research. The first concerns the dynamic specification of the model. In this paper we relied on first order dynamics, but it may be that higher order dynamics would be more appropriate. One strategy is now to find a variety of models with different lag structures and to use likelihood ratio tests to select a final specification. Alternatively, one could develop diagnostic test statistics which could be used to examine remaining residual autocorrelation.

A second topic of further research amounts to extending the approach followed in Chintagunta and Prasad (1998), where interpurchase times are combined with brand choice. Indeed, one could construct models for long-run and short-run effects of marketing mix variables on both marketing performance measures jointly.

Table 1: ML parameter estimates for the static duration model, the common factor model and the error correction duration model, without household heterogeneity. Standard errors appear in parentheses.

	static		common factor		error correction	
			α parameters			
regular price (32 oz.)					1.079	(0.263)
price cut (32 oz.)					−2.278	(0.489)
display					−3.743	(0.690)
line					−0.606	(1.905)
major					−1.412	(0.859)
volume prev. (32 oz.)					0.118	(0.006)
			β parameters			
constant	1.044	(0.095)	1.071	(0.131)	1.063	(0.134)
regular price (32 oz.)	0.606	(0.055)	0.638	(0.076)	0.615	(0.078)
price cut (32 oz.)	−1.761	(0.507)	−2.031	(0.481)	−1.979	(0.842)
display	−2.154	(0.659)	−3.977	(0.665)	−0.378	(1.116)
line	−1.326	(1.984)	−0.346	(1.903)	−4.598	(3.312)
major	−2.692	(0.900)	−1.442	(0.838)	−5.026	(1.595)
volume prev. (32 oz.)	0.130	(0.006)	0.121	(0.006)	0.139	(0.010)
household income	−0.022	(0.004)	−0.023	(0.005)	−0.023	(0.005)
household size	−0.190	(0.008)	−0.194	(0.011)	−0.198	(0.011)
μ_0	0.515	(0.040)	0.496	(0.041)	0.492	(0.042)
δ_0	1.614	(0.049)	1.619	(0.050)	1.610	(0.050)
δ_1	2.013	(0.020)	2.139	(0.022)	2.144	(0.022)
ρ			0.348	(0.012)	0.344	(0.012)
log likelihood	−22 131.57		−21 720.67		−21 707.65	
BIC	5.3855		5.2869		5.2903	
AIC	5.3753		5.2758		5.2741	

Table 2: ML parameter estimates for the static duration model, the common factor and the error correction duration model with household heterogeneity.[1] Standard errors appear in parentheses.

	static		common factor		error correction	
			α parameters			
regular price (32 oz.)					0.862	(0.287)
price cut (32 oz.)					−2.121	(0.599)
display					−4.834	(0.827)
line					0.459	(2.533)
major					−0.375	(1.030)
volume prev. (32 oz.)					0.112	(0.007)
			β parameters			
constant	1.464	(0.176)	1.455	(0.187)	1.412	(0.191)
regular price (32 oz.)	0.594	(0.103)	0.603	(0.109)	0.607	(0.112)
price cut (32 oz.)	−1.831	(0.566)	−1.825	(0.571)	−2.137	(0.778)
display	−4.016	(0.784)	−4.368	(0.777)	−1.894	(1.152)
line	0.327	(2.499)	0.321	(2.458)	−1.785	(3.819)
major	−1.270	(1.016)	−1.013	(1.007)	−2.593	(1.634)
volume prev. (32 oz.)	0.113	(0.006)	0.113	(0.006)	0.124	(0.008)
household income	−0.031	(0.004)	−0.031	(0.005)	−0.034	(0.005)
household size	−0.165	(0.009)	−0.163	(0.010)	−0.164	(0.010)
μ_0	0.188	(0.038)	0.185	(0.038)	0.194	(0.039)
δ_0	1.970	(0.073)	1.964	(0.070)	1.949	(0.073)
δ_1	2.301	(0.028)	2.299	(0.028)	2.307	(0.028)
ρ			0.051	(0.016)	0.046	(0.016)
p_1	0.092	(0.011)	0.563	(0.023)	0.567	(0.023)
p_2	0.350	(0.053)	0.094	(0.013)	0.091	(0.013)
p_3	0.558	(0.079)	0.344	(0.035)	0.342	(0.035)
log likelihood	−21 034.89		−21 021.85		−21 004.38	
BIC	5.1478		5.1479		5.1633	
AIC	5.1154		5.1130		5.1131	

[1] The household heterogeneity is modeled by three mixture components, that is $M = 3$.

[2] The columns display $\hat{\theta} = \sum_{m=1}^{3} \hat{p}_m \hat{\theta}_m$, where θ denotes a general parameter, and standard errors are computed using the delta method, see Greene (1993, p. 297).

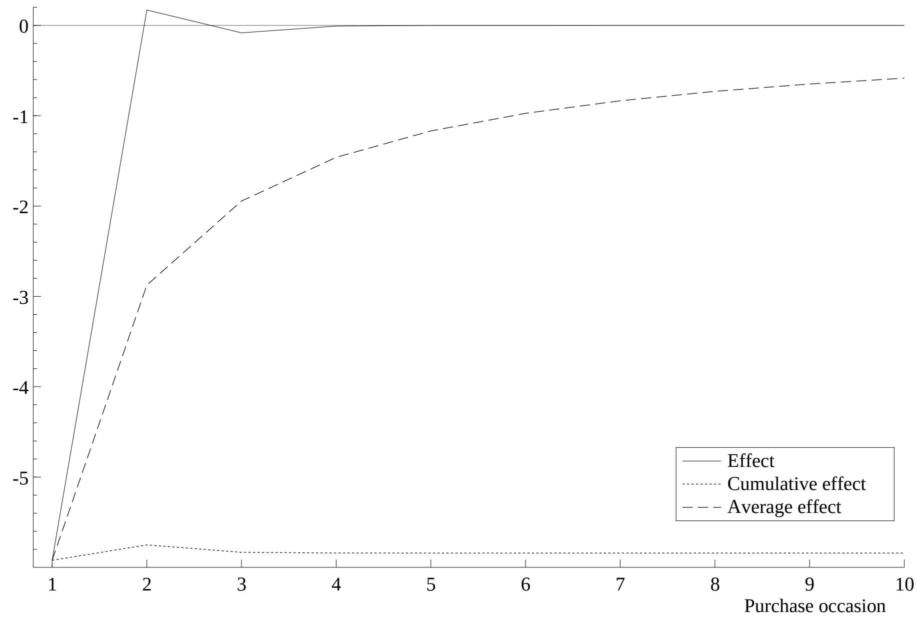

Figure 2: The effect of a unit price cut at $n = 1$ on current and future interpurchase times.

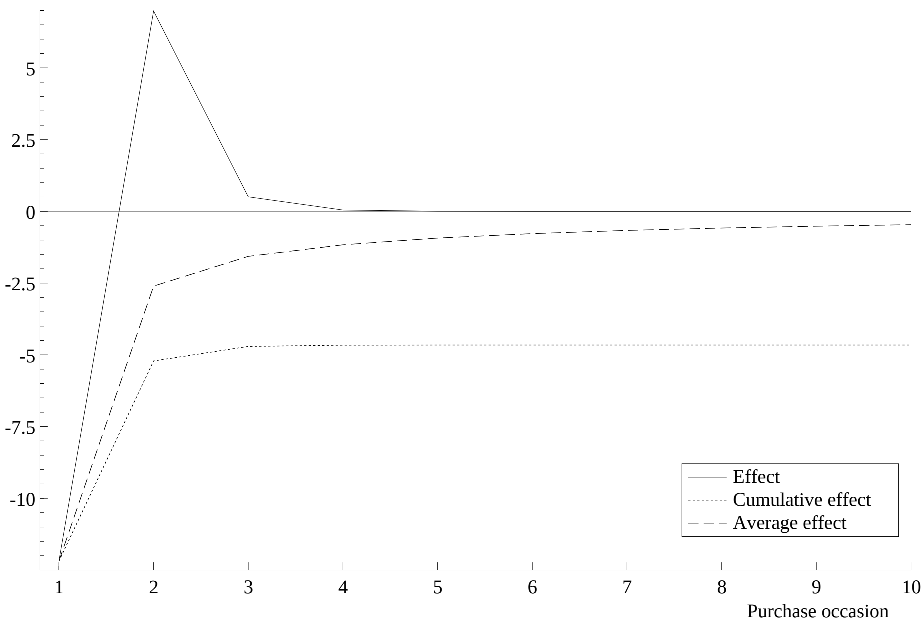

Figure 3: The effect of a display at $n = 1$ on current and future interpurchase times.

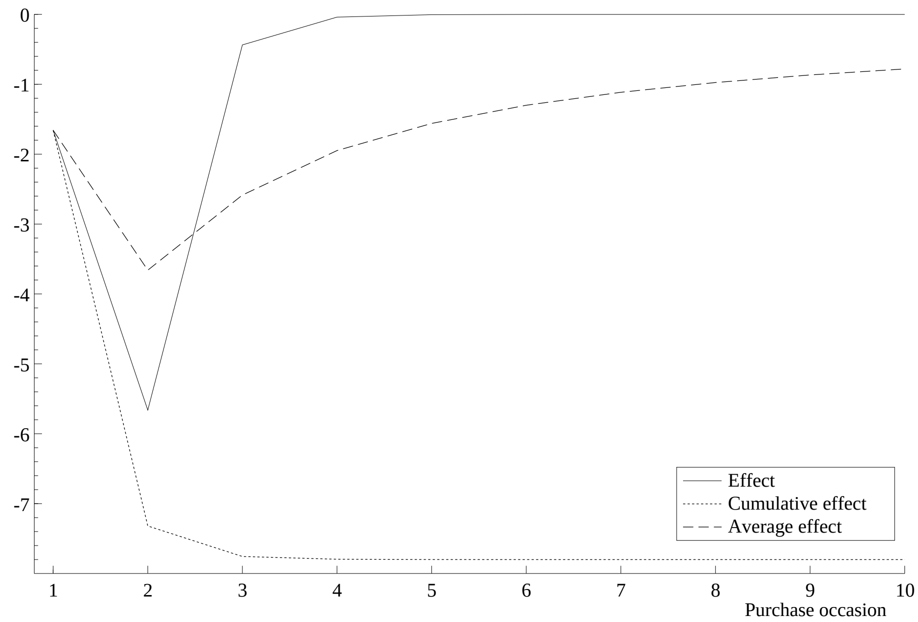

Figure 4: The effect of a major advertisement at $n = 1$ on current and future interpurchase times.

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