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Risk Measures for Autocorrelated Hedge Fund Returns

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ABSTRACT
Standard risk metrics tend to underestimate the true risks of hedge funds because of serial correlation in the reported returns. Getmansky, Lo, and Makarov (2004) derive mean, variance, Sharpe ratio, and beta formulae adjusted for serial correlation. Following their lead, we derive adjusted downside and global measures of individual and systemic risks. We distinguish between normally and fat-tailed distributed returns and show that adjustment is particularly relevant for downside risk measures in the case of fat tails. An empirical analysis reveals that unadjusted risk measures can considerably underestimate the true extent of individual and systemic risks for hedge funds. (*JEL*: G12, G23, G28)

KEYWORDS: hedge funds, Pareto distribution, serial correlation, systemic risk, VaR.

After two decades of strong growth, hedge funds have developed into a mature and widely accepted asset class. On the back of relatively high historical returns the hedge fund industry has enjoyed a near-continuous inflow of new money until the credit crisis. Moreover, hedge fund risk levels are frequently reported to be lower than those of the more traditional investments in equities. These

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performance characteristics of hedge funds have also attracted considerable academic attention.[1]

One particular feature of hedge fund returns is their strong autocorrelation. Fung and Hsieh (2001), Agarwal and Naik (2004), and Huang, Liechty, and Rossi (2012) demonstrate that this feature invalidates standard mean-variance analysis for hedge funds. Getmansky, Lo, and Makarov (2004) argue that the autocorrelation stems from the illiquidity of the assets held by hedge funds and the smoothing of the returns because of reporting practices. Based on a moving average representation of reported returns, they show how this process affects the Sharpe ratio (SR) and beta in a standard single-factor model. As the smoothing lowers the variance and the covariance (with the market index) but leaves the mean unaffected, the standard risk measures tend to underestimate the actual risk (SR is overstated). In Chan et al. (2006), this framework is used to evaluate the systemic risk posed by hedge funds for the banking sector. Bollen and Pool (2009) use this autocorrelation structure to detect misreporting. Recently, Avramov et al. (2011) and Teo (2011) use the algorithm of Getmansky, Lo, and Makarov (2004) to unsmooth hedge fund returns.

This article extends the lead taken by Getmansky, Lo, and Makarov (2004) in two dimensions. First, Getmansky, Lo, and Makarov (2004) consider two measures of risk (SR and market beta) that can be defined as “global”, because all returns are used to calculate them. We broaden their paper's scope by evaluating three downside measures of risk, two univariate, and one multivariate (or systemic), that use only (part of) negative returns. There is considerable evidence from behavioral finance that individuals do not treat the upside potential and the downside risk symmetrically. Moreover, regulatory frameworks such as Solvency II and Basel III focus on downside risk measures.

The second direction adds the distinction between light tails and heavy tails. The measures considered by Getmansky, Lo, and Makarov (2004) fully characterize the risk aspects in the case that the noise is multivariate normally distributed, that is, in the case of light tails. In practice, it is known that return distributions of most assets are heavy tailed. An example of a heavy-tailed distribution is the Student's *t*-distribution. Such distributions exhibit hyperbolic or power-like decline in the tails, whereas light-tailed distributions have exponential declining tails. Whereas the SR and beta measures also apply in case of heavy tails (as long as second moments are finite), the downside risk measures do respond quite differently to smoothing depending on whether the returns are light or heavy tailed.

More specifically, apart from the univariate global SR measure considered by Getmansky, Lo, and Makarov (2004), we also investigate the value-at-risk (VaR) and expected shortfall (ES) measures.[2] The VaR and ES downside measures play a

[1]For research on the risk and return characteristics of hedge funds, see Agarwal and Naik (2004); Morton, Popova, and Popova (2006); Fung et al. (2008); Sadka (2010); Dichev and Yu (2011); and Huang, Liechty, and Rossi (2012).

[2]We acknowledge that several of the performance metrics in this article are criticized (see e.g., Lo 2002 or Goetzmann et al. 2007). These measures can be gamed and have shortcomings when funds trade

central role in the risk management practices of the financial sector and are also sensitive to the type of tail behavior (light or fat) of the returns under consideration. As for the multivariate risk measures, we examine the correlation coefficient ρ, which is a global risk measure, and a multivariate measure that focuses on the downside systemic risk. The latter measure reflects the amount of interdependence between two or more returns deep into the joint tail loss area. It exclusively picks up the extreme linkages in crisis situations. This measure we term as the extreme linkage measure (ELM); it is explained in Section 1.2.2. The chart below summarizes our investigation for both light and heavy tails.

Risk measures analyzed in the article

	Scope	
Dimension	Global	Downside
Individual	**SR**	**VaR, ES**
Systemic	$\boldsymbol{\rho}$	**ELM**

After investigating how these risk measures are affected by the kind of smoothing proposed by Getmansky, Lo, and Makarov (2004), these smoothing-adjusted risk measures are applied on two broad-based hedge fund indices for the period between 1990 and 2013. We find that the smoothing-adjusted hedge fund investment returns indicate levels of risk that can be considerably higher than the risk measures based on reported returns. This finding applies in particular for the downside risk measures.

Using the smoothing-adjusted economic risk measures is important for both investors trying to determine the proportion to invest in hedge funds and for investors constructing a hedge fund portfolio based on the relative risks of those funds. Correct risk measures are instrumental in preventing overpaying for an investment in hedge funds because its attractiveness has been overestimated. Finally, the ELM results can be of interest for policy makers and regulators who are concerned about the possible effects of hedge funds on financial stability.

The article proceeds as follows. Section 1 models the impact of smoothing and derives the adjusted risk measures. Section 2 presents the empirical methodology. In Section 3, the adjusted risk measures are applied to two hedge fund indices. Section 4 concludes. For the sake of brevity, several details are provided in a separate web appendix (Di Cesare, Stork, and de Vries 2014).

nonlinear contracts or engage in market-timing strategies. Alternative metrics exist. The purpose of this article, however, is to analyze the existing framework of widely used performance measures and to extend the work of Getmansky, Lo, and Makarov (2004). The analysis of other metrics is outside the scope of this article.

1 MODELING THE IMPACT OF SMOOTHING

We derive how smoothing affects the risk measures. Following Getmansky, Lo, and Makarov (2004), the reported or observed returns are considered to be a weighted average of the fund's actual returns over a number of the most recent periods, including the current period. This assumption turns the observed returns into a moving average of the actual returns. Consider two hedge funds (indices) with actual returns $X_{1,t}$ and $X_{2,t}$ in period t, which are independent and identically distributed (i.i.d.) through time (but they may be correlated with each other). The actual returns cannot be observed directly and the reported returns $S_{i,t}$ are governed by the following MA(K) process

$$S_{i,t}=\sum_{k=0}^{K}\theta_{i,k}X_{i,t-k}, \tag{1}$$

$$\theta_{i,k}\in[0,1], \quad k=0,\ldots,K, \tag{2}$$

$$\sum_{k=0}^{K}\theta_{i,k}=1, \tag{3}$$

for $i=1,2$. We refer to the MA coefficients $\theta_{i,k}$ as the smoothing coefficients.

The model in Equations (1) to (3) is very general but rich enough to study the impact of smoothing on individual risk measures.[3] To analyze the consequences on systemic risk measures we introduce the source of dependence by means of the single-factor model. Thus, if R_t is the market return in period t and if $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are idiosyncratic risk factors in period t, then

$$X_{i,t}=\beta_i R_t+\varepsilon_{i,t}, \tag{4}$$

for $i=1,2$. For the sake of the presentation, we assume that both β_1 and β_2 are strictly positive. Moreover, R_t, $\varepsilon_{1,t}$, and $\varepsilon_{2,t}$ are i.i.d., with distributions that are specified below. The model can be generalized to the case in which there are multiple market factors.[4]

1.1 Smoothing Effects on Univariate Risk Measures

We study the VaR and ES measures in the case of light tails and heavy tails. The light tail case is focused on the normal distribution, which is the standard fare in finance. It provides a benchmark against which the case of heavy tails is judged. We first clarify the concept of a heavy tail.

[3] We refer to Getmansky, Lo, and Makarov (2004, p. 545–547) for a detailed exposition on why an MA(K) model is appropriate

[4] The corresponding equations are reported in Section 1 of Di Cesare, Stork, and de Vries (2014).

A distribution is said to be (symmetrically) heavy tailed if it is regularly varying at infinity, that is to say, the tails of the distribution satisfy

$$\lim_{t\to\infty}\frac{F(-tx)}{F(-t)}=\lim_{t\to\infty}\frac{1-F(tx)}{1-F(t)}=x^{-\alpha}, \tag{5}$$

for all $x>0$ and some $\alpha>0$. The tail index α determines how heavy the tails are, as only the moments upto α are bounded. For example, it is readily verified that the Student's t-distribution with v degrees of freedom has regularly varying tails with $\alpha=v$. Moreover, we will assume that the following first-order expansion applies

$$\mathbb{P}(X>x)=Ax^{-\alpha}+o\left(x^{-\alpha}\right), \tag{6}$$

where $A>0$. The A is not necessarily constant but can be a slowly varying function, that is, $\lim_{t\to\infty}A(tx)/A(t)=1$ for any $x>0$. Somewhat loosely formulated, this definition means that to a first order at infinity the distribution follows a Pareto distribution. Many popular distributions satisfy Equations (5) and (6), such as the Student-t and infinite variance sum stable distributions.

To derive the implications of the MA(K) process for our risk measures, we need to know the distribution of the convolution of the random variables in Equations (1) to (3). How to do this for the normal distribution is commonly known as the square root rule. For example, assuming that the market factor $R\sim N(\mu_R,\sigma_R)$ and the idiosyncratic factor $\varepsilon\sim N(\mu_\varepsilon,\sigma_\varepsilon)$ in Equation (4), it follows (omitting subindices i whenever there is no possible confusion)

$$X\sim N\left(\beta\mu_R+\mu_\varepsilon,\left(\beta^2\sigma_R^2+\sigma_\varepsilon^2\right)^{1/2}\right). \tag{7}$$

Such a simple rule does not generally exist for the case of heavy-tailed distributions. For our purposes, however, it suffices to know what happens to the tail probabilities under summation, which is a considerably simpler problem. To derive our results, we make use of the celebrated convolution theorem by Feller (1971, ch. VIII.8). The flavor of the convolution theorem is demonstrated for the case of the single-factor model from Equation (4). Suppose, for example, that both the market factor R and the idiosyncratic risk ε have a Student's t-distribution with v degrees of freedom and hence satisfy Equation (6). It then holds that

$$\lim_{t\to\infty}\frac{\mathbb{P}(X>t)}{(\beta^v+1)At^{-v}}=1. \tag{8}$$

It transpires that one can just add the marginal tail probabilities. To a first-order, sufficiently deep into the tail all probability mass in the plane concentrates along the axes and this property determines the sum. If the tail indices are unequal, then the tail with the lowest index dominates the sum. Note that the convolution changes the scale factor, but leaves the power v unaffected. In other words, if one studies the convolution of two independent heavy-tail distributed random variables with the same tail index at large quantiles, it suffices to take the sum of the scales divided by the quantile to the power of the tail index.

With these preparations at hand, we turn to the specific measures and investigate the effects of the moving average feature of reported returns. Getmansky, Lo, and Makarov (2004) already find that SR is overstated when hedge fund returns are smoothed (provided the variance is finite). Next, we assess the downside risk measures VaR and ES.

1.1.1 Value-at-risk. VaR is probably the most widely used univariate measure in risk management (see, e.g., Jorion 2000), so it is worth studying how it is affected by the presence of autocorrelation in hedge fund returns.

For the random variable Y_t with a continuous distribution, the VaR at the confidence level $1-p$ is defined implicitly by

$$\mathbb{P}\big(Y_t > \mathrm{VaR}(Y_t,p)\big) = p, \quad \text{or} \quad \mathrm{VaR}(Y_t,p) = \varphi_{Y_t}^{-1}(1-p), \tag{9}$$

where $\varphi_{Y_t}^{-1}(x)$ is the inverse of the cumulative density function of Y_t evaluated at x. Note that the VaR usually is a loss return and hence a positive number. Therefore, we focus on the right tail of the loss distribution.

1.1.1.1 Normal distribution. Let $\mathrm{VaR}(S_t,p;N)$ denote the VaR at the confidence level $1-p$ for the reported return S_t. Under the assumption that the actual returns X_t follow the normal distribution $X_t \sim N(\mu_X,\sigma_X)$, we have from Equations (1) to (3) that $S_t \sim N(\mu_S,\sigma_S)$, where

$$\mu_S = \left(\sum_{k=0}^{K}\theta_k\right)\mu_X = \mu_X \quad \text{and} \quad \sigma_S = \left(\sum_{k=0}^{K}\theta_k^2\right)^{1/2}\sigma_X \le \sigma_X. \tag{10}$$

Since

$$p = \mathbb{P}\big(S_t > \mathrm{VaR}(S_t,p;N)\big) = 1-\Phi\left(\frac{\mathrm{VaR}(S_t,p;N)-\mu_S}{\sigma_S}\right), \tag{11}$$

where $\Phi(x)$ is the standard normal cumulative distribution function evaluated at x, the VaR is given by

$$\mathrm{VaR}(S_t,p;N) = \sigma_S\Phi^{-1}(1-p)+\mu_S. \tag{12}$$

Given that $\mu_S=\mu_X$ and $\sigma_S \le \sigma_X$, the VaR calculated on the smoothed returns, $\mathrm{VaR}(S_t,p;N)$, is always smaller than or equal to the VaR calculated on the actual returns,

$$\mathrm{VaR}(X_t,p;N) = \sigma_X\Phi^{-1}(1-p)+\mu_X. \tag{13}$$

In particular, from Equations (10), (12), and (13) we have the following:

Proposition 1: *If the actual returns X_t are i.i.d. with a normal distribution, then the VaR of the reported returns is related to the VaR of the actual returns as follows*

$$\frac{\mathrm{VaR}(S_t,p;N)-\mu_S}{\mathrm{VaR}(X_t,p;N)-\mu_S}=\frac{\sigma_S}{\sigma_X}=\left(\sum_{k=0}^{K}\theta_k^2\right)^{1/2}. \tag{14}$$

Under the assumption of a normal distribution, the presence of autocorrelation in the actual hedge fund returns reduces the reported VaR by the reduction in the volatility of the returns. The square root rule applies again.

1.1.1.2 Heavy tails. Suppose now that the distribution of X_t is heavy tailed as in Equation (6) and let $\mathrm{VaR}(S_t,p;H)$ denote the VaR at confidence level $1-p$ for the case of heavy-tailed distributions. Invoking Feller's convolution theorem gives

$$\lim_{p\downarrow 0}\frac{\mathbb{P}\big(S_t>\mathrm{VaR}(S_t,p;H)\big)}{\left(\sum_{k=0}^{K}\theta_k^\alpha\right)\gamma_X\mathrm{VaR}(S_t,p;H)^{-\alpha}}=1, \tag{15}$$

where γ_X is the scale factor of X_t (i.e., the parameter A in Equation 6). Upon first-order inversion, for small p approximately

$$\mathrm{VaR}(S_t,p;H)\simeq\left(\frac{\gamma_X}{p}\sum_{k=0}^{K}\theta_k^\alpha\right)^{1/\alpha}. \tag{16}$$

Similarly, one shows that for the actual returns

$$\mathrm{VaR}(X_t,p;H)\simeq\left(\frac{\gamma_X}{p}\right)^{1/\alpha}. \tag{17}$$

Note that $\gamma_X\sum_k\theta_k^\alpha$ is the scale factor of S_t, which can be estimated from the data. Combining Equations (16) and (17) we obtain the following:

Proposition 2: *If the actual returns X_t are i.i.d. with heavy tails as in Equation* (6)*, then*

$$\lim_{p\downarrow 0}\frac{\mathrm{VaR}(S_t,p;H)}{\mathrm{VaR}(X_t,p;H)}=\left(\sum_{k=0}^{K}\theta_k^\alpha\right)^{1/\alpha}. \tag{18}$$

Given that Equations (2) and (3) imply $\sum_k\theta_k^\alpha\le 1$ for any $\alpha\ge 1$, the VaR calculated on smoothed returns deep into the tail area is always smaller than or equal to the VaR calculated on actual returns. The latter condition just requires that the mean is bounded. In case of the Cauchy distribution, which has $\alpha = 1$, the VaR calculated on smoothed returns is equal to the VaR calculated on actual returns.

As the first derivative of $(\sum_k \theta_k^\alpha)^{1/\alpha}$ with respect to the tail index α is negative, the presence of autocorrelation affects the smoothed returns' VaR relatively less when the reported return distribution has fatter tails (smaller α). In general, for any given value of the tail index $\alpha > 1$, the VaR of the smoothed returns $\mathrm{VaR}(S_t, p; H)$ is minimized when the smoothing coefficients θ_k equal $1/(K+1)$ for all k. In this case, the current and past true economic returns are equally weighted and together make up the reported returns. The ratio of $\mathrm{VaR}(S_t, p; H)$ to $\mathrm{VaR}(X_t, p; H)$ equals $(K+1)^{\frac{1-\alpha}{\alpha}}$. For $K = 2$ and $\alpha = 3$ this finding implies, for instance, that the reported VaR could be equal to less than half of the true VaR.

Finally, note that the correction term in Equation (18) for the heavy tail case is always smaller than the correction term in Equation (14) for the normal case as long as $\alpha > 2$ (i.e., finite variance). In other words, the impact of smoothing is usually larger in the heavy-tail case than in the normal case.

1.1.2 Expected shortfall.

For a given loss return threshold y, the ES measure is the conditional expectation

$$\mathrm{ES}(Y_t, y) = \mathbb{E}[Y_t | Y_t > y]. \tag{19}$$

In general, the ES measure is difficult to compute for the convolution induced by the MA(K) process. Fortunately, for the normal case and the heavy tail case at sufficiently large y, we have explicit results.

1.1.2.1 Normal distribution.

Assume that actual returns X_t have a normal distribution $X_t \sim N(\mu_X, \sigma_X)$, then ES equals

$$\mathrm{ES}(S_t, y; N) = \frac{\int_y^\infty x e^{-\frac{1}{2}\left(\frac{x-\mu_S}{\sigma_S}\right)^2} \mathrm{d}x}{\int_y^\infty e^{-\frac{1}{2}\left(\frac{x-\mu_S}{\sigma_S}\right)^2} \mathrm{d}x} = \frac{\sigma_S \phi\left(\frac{y-\mu_S}{\sigma_S}\right)}{1 - \Phi\left(\frac{y-\mu_S}{\sigma_S}\right)} + \mu_S. \tag{20}$$

It can be shown that the first derivative of ES in Equation (20) with respect to σ_S is positive so that, given that $\sigma_S \leq \sigma_X$, the ES calculated on smoothed returns is always lower than or equal to the ES calculated on actual returns. Furthermore, when y equals the VaR that corresponds to confidence level $1-p$, we derive

$$\begin{aligned} \mathrm{ES}(S_t, \mathrm{VaR}(S_t, p; N); N) &= \frac{\sigma_S e^{-\frac{1}{2}\left(\frac{\mathrm{VaR}(S_t, p; N) - \mu_S}{\sigma_S}\right)^2}}{p\sqrt{2\pi}} + \mu_S \\ &= \frac{e^{-\frac{1}{2}(\Phi^{-1}(1-p))^2}}{p\sqrt{2\pi}\,\Phi^{-1}(1-p)} \left(\mathrm{VaR}(S_t, p; N) - \mu_S\right) + \mu_S. \end{aligned} \tag{21}$$

The last equation shows that the ES is proportional to the VaR:

Proposition 3: *If the actual returns X_t are i.i.d. with a normal distribution, then ES of the reported returns is related to ES of the actual returns as follows*

$$\frac{\mathrm{ES}(S_t,\mathrm{VaR}(S_t,p;N);N)-\mu_S}{\mathrm{ES}(X_t,\mathrm{VaR}(X_t,p;N);N)-\mu_S}=\frac{\sigma_S}{\sigma_X}=\left(\sum_{k=0}^{K}\theta_k^2\right)^{1/2}. \tag{22}$$

As in the case of VaR, we find that after the correction for the mean, the smoothed ES is proportional to the actual ES. The presence of autocorrelation in the reported hedge fund returns reduces the estimated ES, and the reduction is proportional to the ratio of the two volatility estimates.

1.1.2.2 Heavy tails. Under the same assumptions as for the VaR, we have that

$$\lim_{x\to\infty}\frac{\mathbb{P}(S_t>x)}{\left(\sum_{k=0}^{K}\theta_k^\alpha\right)\gamma_X x^{-\alpha}}=1. \tag{23}$$

If the distribution function is monotonic in the tail area, it holds furthermore that the density satisfies the following asymptotic expansion (see Bingham, Goldie, and Teugels 1987)

$$\lim_{x\to\infty}\frac{f_{S_t}(x)}{\alpha\left(\sum_{k=0}^{K}\theta_k^\alpha\right)\gamma_X x^{-\alpha-1}}=1. \tag{24}$$

Hence, for a sufficiently large threshold y and if $\alpha>1$, the ES measure is approximately equal to

$$\mathrm{ES}(S_t,y;H)\simeq\frac{\alpha\left(\sum_{k=0}^{K}\theta_k^\alpha\right)\gamma_X\int_y^\infty x x^{-\alpha-1}\,\mathrm{d}x}{\left(\sum_{k=0}^{K}\theta_k^\alpha\right)\gamma_X y^{-\alpha}}=\frac{\alpha}{\alpha-1}y. \tag{25}$$

Equation (25) shows that the ES is independent of the smoothing coefficients θ_k. The reason for this independence is that the smoothing coefficients affect the expected value of the exceedances and the probability of exceeding the threshold in the same proportion. As a result, both effects cancel out. The ES is therefore invariant to smoothing of the returns in the heavy tail case.

For $y=\mathrm{VaR}(S_t,p;H)$, we obtain from Equation (25) that

$$\mathrm{ES}(S_t,\mathrm{VaR}(S_t,p;H);H)\simeq\frac{\alpha}{\alpha-1}\mathrm{VaR}(S_t,p;H). \tag{26}$$

The properties of the ES exactly match those of the VaR. In particular, we have:

Proposition 4: *If the actual returns X_t are i.i.d. with heavy tails as in Equation* (6), *then*

$$\lim_{p\downarrow 0}\frac{\mathrm{ES}(S_t,\mathrm{VaR}(S_t,p;H);H)}{\mathrm{ES}(X_t,\mathrm{VaR}(X_t,p;H);H)}=\left(\sum_{k=0}^{K}\theta_k^{\alpha}\right)^{1/\alpha}. \tag{27}$$

This last result shows that ES calculated on smoothed returns is always smaller than or equal to ES calculated on actual returns. This result mimics the one found for the VaR metric in Equation (18).

To conclude, both in the case of normally distributed returns and in case of heavy tails, the VaR and ES measures are proportional to each other and are similarly affected by the smoothing because of reporting.

1.2 Smoothing Effects on Multivariate Risk Measures

In this subsection, two systemic risk measures are investigated. Getmansky, Lo, and Makarov (2004) already consider how the estimate of the market beta for a single-factor model is reduced due to smoothing. As their results apply for the normal case and the heavy tail case as long as $\alpha > 2$, those results are not reproduced here. Instead, we focus on the correlation coefficient ρ and the downside systemic risk measure ELM.

1.2.1 Pairwise correlation. We investigate the effects of smoothing on the correlation between the reported returns $(S_{1,t}, S_{2,t})$ of two hedge funds under the assumption that actual returns $(X_{1,t}, X_{2,s})$ have a correlation equal to $\rho(X_1, X_2)$ if $t = s$ and zero otherwise. Moreover, the variances of the actual returns are given by σ_1 and σ_2. In this general framework, only the second moments of the returns are required to exist. Standard results and the Cauchy–Schwarz inequality imply:

Proposition 5: *Assume that the actual returns $(X_{1,t}, X_{2,s})$ have bounded second moments and correlation equal to $\rho(X_1, X_2)$ if $t = s$ and zero otherwise, then the correlation coefficient of the reported returns is related to the correlation coefficient of the actual returns as follows*

$$\rho(S_1,S_2)=\frac{\sum_{k=0}^{K}\theta_{1,k}\theta_{2,k}}{\left(\left(\sum_{k=0}^{K}\theta_{1,k}^2\right)\left(\sum_{k=0}^{K}\theta_{2,k}^2\right)\right)^{1/2}}\rho(X_1,X_2)\leq\rho(X_1,X_2). \tag{28}$$

Note that the correlation calculated on reported returns $\rho(S_1, S_2)$ equals the correlation calculated on actual returns $\rho(X_1, X_2)$ when $\theta_{1,k} = \theta_{2,k}$ for all k, that is when the actual returns of the two hedge funds show exactly the same pattern of autocorrelation. Except for this exceptional case, the correlation calculated on reported returns underestimates the true correlation calculated on actual returns.[5]

1.2.2 The extreme linkage measure.

The ELM is a nonparametric measure of dependence based on extreme value theory (EVT). It was introduced by Huang (1992) and has been applied in several empirical studies of systemic risk (see, e.g., Hartmann, Straetmans, and de Vries 2004; Straetmans, Verschoor, and Wolff 2008). The ELM is defined as the probability that two hedge funds face losses above a threshold s, given that at least one of these faces a loss in excess of that same threshold s

$$\mathrm{ELM}(S_{1,t}, S_{2,t}; s) = \frac{\mathbb{P}(S_{1,t} > s, S_{2,t} > s)}{1 - \mathbb{P}(S_{1,t} \leq s, S_{2,t} \leq s)}. \tag{29}$$

For theoretical purposes, the ELM is evaluated in the limit as s tends to infinity

$$\mathrm{ELM}(S_{1,t}, S_{2,t}) = \lim_{s \to \infty} \mathrm{ELM}(S_{1,t}, S_{2,t}; s). \tag{30}$$

EVT then shows that the value obtained has relevance at finite levels, as long as s is very large, since the tail shape of the distribution approaches the Pareto term in Equation (6) in a smooth manner (see, Balkema and de Haan 1974).

The ELM also indicates the expected number of hedge funds that are stressed, n, given that at least one of the hedge funds is stressed, minus one

$$\begin{aligned}\mathbb{E}[n | n \geq 1] &= \frac{\mathbb{P}(S_{1,t} > s) + \mathbb{P}(S_{2,t} > s)}{1 - \mathbb{P}(S_{1,t} \leq s, S_{2,t} \leq s)} \\ &= 1 + \mathrm{ELM}(S_{1,t}, S_{2,t}; s).\end{aligned} \tag{31}$$

In fact, the conditional expectation measure $\mathbb{E}[n|n \geq 1]$ also has the advantage that it can be easily extended to higher dimensions ($n > 2$).

1.2.2.1 Normal distribution.

Assume that $X_{1,t}$ and $X_{2,t}$ are multivariate normally distributed with correlation ρ and standard deviations σ_1 and σ_2. To derive ELM,

[5]If the smoothing coefficients were allowed to be negative, there could also be the extreme case in which $\sum_{k=0}^{K} \theta_{1,k}\theta_{2,k} = 0$, so that $\rho(S_{1,t}, S_{2,t}) = 0$ irrespective of the value of $\rho(X_{1,t}, X_{2,t})$. Hence, reported returns would always appear to be uncorrelated even if, for example, the correlation between the actual returns were equal to one.

we adopt the proof of Sibuya (1960). Note that, by elementary manipulations

$$
\begin{aligned}
1+\mathrm{ELM}(X_{1,t},X_{2,t};s) &= \frac{\mathbb{P}(X_{1,t}>s)+\mathbb{P}(X_{2,t}>s)}{1-\mathbb{P}(X_{1,t}\leq s, X_{2,t}\leq s)} \\
&= \frac{1}{1-\frac{\mathbb{P}(X_{1,t}>s,X_{2,t}>s)}{\mathbb{P}(X_{1,t}>s)+\mathbb{P}(X_{2,t}>s)}} \\
&\leq \frac{1}{1-\frac{\mathbb{P}(X_{1,t}+X_{2,t}>2s)}{\mathbb{P}(X_{1,t}>s)+\mathbb{P}(X_{2,t}>s)}},
\end{aligned}
\tag{32}
$$

as the line $X_{1,t}+X_{2,t}=2s$ cuts the plane $(X_{1,t}>s, X_{2,t}>s)$ from below. Note that $\big(X_{1,t}+X_{2,t}\big)/2$ has variance $\big(\sigma_1^2+\sigma_2^2+2\sigma_1\sigma_2\rho\big)/4$, which is strictly smaller than $\max(\sigma_1^2,\sigma_2^2)$ as long as $\rho\neq 1$ or $\sigma_1\neq\sigma_2$.

The classical Laplace tail expansion of a standard normal distribution $\Phi(s)$ with density $\phi(s)$ holds that $1-\Phi(s)\simeq\phi(s)/s$ for large s. It then follows that

$$
\lim_{s\to\infty}\frac{\mathbb{P}\big((X_{1,t}+X_{2,t})/2>s\big)}{\mathbb{P}(X_{1,t}>s)+\mathbb{P}(X_{2,t}>s)}=0,
\tag{33}
$$

as the rate of the exponential decay of the density of the sum (divided by two), dictated by the inverse of its variance, is greater than the rate of the exponential decay of at least one of the individual probabilities (one speaks of asymptotic independence). Hence, $\mathrm{ELM}(X_{1,t},X_{2,t})=0$. As the proof does not depend on the particular values of the correlation and variances, it immediately follows that $\mathrm{ELM}(S_{1,t},S_{2,t})=0$ as well, as long as $\rho(S_1,S_2)\neq 1$ or the variances of the reported returns of the two hedge funds are different. Note that Equation (28) implies that the correlation between smoothed returns is always smaller than one when $\theta_{1,k}\neq\theta_{2,k}$ for at least one k. In summary:

Proposition 6: *If actual returns $(X_{1,t},X_{2,t})$ are i.i.d. and normally distributed, with correlation ρ and standard deviations σ_1 and σ_2, then*

$$
\mathrm{ELM}(X_{1,t},X_{2,t})=
\begin{cases}
1\,\textit{and}\,\mathrm{ELM}(S_{1,t},S_{2,t})=1, & \textit{if}\ \rho=1,\sigma_1=\sigma_2,\\
& \quad \theta_{1,k}=\theta_{2,k}\ \textit{for all}\ k\\
1\,\textit{and}\,\mathrm{ELM}(S_{1,t},S_{2,t})=0, & \textit{if}\ \rho=1,\sigma_1=\sigma_2,\\
& \quad \theta_{1,k}\neq\theta_{2,k}\ \textit{for some}\ k\\
0\,\textit{and}\,\mathrm{ELM}(S_{1,t},S_{2,t})=0, & \textit{otherwise.}
\end{cases}
\tag{34}
$$

For light tails, the ELM equals zero for both actual and smoothed returns in all relevant cases, and thus is uninformative. This outcome is in sharp contrast with the case of heavy tails.

The ELM is not affected in case the copula of the joint distribution of variables $S_{1,t}$ and $S_{2,t}$ implies asymptotic independence. That result holds for the multivariate normal, but also for multivariate exponential distributions. Per contrast, as we show below, if there is asymptotic dependence, the ELM may be affected. Insofar as the systemic risk measure presumes thin tails, for example, normality of returns, systemic risk in the sense of multiple crashes occurring at the same time are very unlikely. For example, LTCM had a risk management system built on the multivariate normal distribution and hence movements in multiple markets were attributed a zero probability. Nevertheless, the Russian and Asian crises combined into one event. We refer to Jorion (2000) for an excellent discussion of the LTCM case, and the hazardous assumption of multivariate normal returns.

1.2.2.2 Heavy tails. In addition to the model given by Equations (1) to (3) we now make the further assumption that the single-factor model in Equation (4) applies. This is a natural way to introduce dependence between two hedge fund returns.

Assume that R_t, $\varepsilon_{1,t}$, and $\varepsilon_{2,t}$ are heavy tailed, so that

$$\lim_{s\to\infty}\frac{\mathbb{P}(R_t>s)}{\gamma_R s^{-\alpha}}=\lim_{s\to\infty}\frac{\mathbb{P}(\varepsilon_{1,t}>s)}{\gamma_{\varepsilon_1} s^{-\alpha}}=\lim_{s\to\infty}\frac{\mathbb{P}(\varepsilon_{2,t}>s)}{\gamma_{\varepsilon_2} s^{-\alpha}}=1, \tag{35}$$

where the scale parameters γ_R, γ_{ε_1}, γ_{ε_2} are strictly positive constants. For values of the threshold s, high enough such that Feller's theorem provides a good approximation for the convolution of the random variables, we have

$$\mathbb{P}(X_{i,t}>s)\simeq(\beta_i^\alpha\gamma_R+\gamma_{\varepsilon_i})s^{-\alpha}, \tag{36}$$

$$1-\mathbb{P}(X_{1,t}\le s,X_{2,t}\le s)\simeq\left(\gamma_{\varepsilon_1}+\gamma_{\varepsilon_2}+(\max(\beta_1,\beta_2))^\alpha\gamma_R\right)s^{-\alpha}. \tag{37}$$

The first expression in Equation (36) is a straightforward application of Feller's theorem as explained in Equation (8). For the second expression in Equation (37), notice that the two idiosyncratic risk factors and the market risk comprise the three independent univariate random variables that span the space of X_1 and X_2. The boundary of $1-\mathbb{P}(X_{1,t}\le s,X_{2,t}\le s)$ is a pyramid-shaped figure with respective boundaries of $\gamma_{\epsilon_1}s^{-\alpha}$, $\gamma_{\epsilon_2}s^{-\alpha}$, and $\max(\beta_1,\beta_2)^\alpha\gamma_R s^{-\alpha}$. Summation then yields the right-hand side of Equation (37).

From Equation (31) we have

$$\begin{aligned}\mathrm{ELM}(X_{1,t},X_{2,t})&=\frac{\beta_1^\alpha\gamma_R+\gamma_{\varepsilon_1}+\beta_2^\alpha\gamma_R+\gamma_{\varepsilon_2}}{\gamma_{\varepsilon_1}+\gamma_{\varepsilon_2}+\left(\max(\beta_1,\beta_2)\right)^\alpha\gamma_R}-1\\&=\frac{\left(\min(\beta_1,\beta_2)\right)^\alpha\gamma_R}{\gamma_{\varepsilon_1}+\gamma_{\varepsilon_2}+\left(\max(\beta_1,\beta_2)\right)^\alpha\gamma_R}.\end{aligned} \tag{38}$$

For the actual hedge fund returns $(S_{1,t},S_{2,t})$ the following equations apply:

$$\mathbb{P}(S_{i,t}>s)=\mathbb{P}(X_{i,t}>s)\sum_{k=0}^{K}\theta_{i,k}^\alpha=(\beta_i^\alpha\gamma_R+\gamma_{\varepsilon_i})s^{-\alpha}\sum_{k=0}^{K}\theta_{i,k}^\alpha, \tag{39}$$

$$1-\mathbb{P}(S_{1,t}\le x, S_{2,t}\le x)=\sum_{k=0}^{K}\Big(\theta_{1,k}^{\alpha}\gamma_{\varepsilon_1}+\theta_{2,k}^{\alpha}\gamma_{\varepsilon_2}+\big(\max(\beta_1\theta_{1,k},\beta_2\theta_{2,k})\big)^{\alpha}\gamma_R\Big)s^{-\alpha}, \quad (40)$$

so that

$$\begin{aligned}\mathrm{ELM}(S_{1,t},S_{2,t})&=\frac{(\beta_1^{\alpha}\gamma_R+\gamma_{\varepsilon_1})\sum_{k=0}^{K}\theta_{1,k}^{\alpha}+(\beta_2^{\alpha}\gamma_R+\gamma_{\varepsilon_1})\sum_{k=0}^{K}\theta_{2,k}^{\alpha}}{\sum_{k=0}^{K}\big(\theta_{1,k}^{\alpha}\gamma_{\varepsilon_1}+\theta_{2,k}^{\alpha}\gamma_{\varepsilon_2}+\big(\max(\beta_1\theta_{1,k},\beta_2\theta_{2,k})\big)^{\alpha}\gamma_R\big)}-1\\&=\frac{\sum_{k=0}^{K}\big(\min(\beta_1\theta_{1,k},\beta_2\theta_{2,k})\big)^{\alpha}\gamma_R}{\sum_{k=0}^{K}\big(\theta_{1,k}^{\alpha}\gamma_{\varepsilon_1}+\theta_{2,k}^{\alpha}\gamma_{\varepsilon_2}+\big(\max(\beta_1\theta_{1,k},\beta_2\theta_{2,k})\big)^{\alpha}\gamma_R\big)}.\end{aligned} \quad (41)$$

Note that Equation (41) simplifies to Equation (38) when both of the hedge funds' actual returns are smoothed in exactly the same way, that is, when $\theta_{1,k}=\theta_{2,k}$ for all k. In this case, as in the pairwise correlation, the measure of linkage calculated on reported returns is equal to that calculated on actual returns.

The previous sections show that the presence of autocorrelation reduces SR, VaR, ES, and pairwise correlation, both in the case of normally distributed returns and in case of heavy tails. In the case of heavy tails, we have the following corresponding result for ELM (proven in Appendix A.1):

Proposition 7: *Suppose that actual hedge fund returns* $(X_{1,t},X_{2,t})$ *have the same market exposure as* $\beta_1=\beta_2=\beta$. *Moreover, reported returns* $(S_{1,t},S_{2,t})$ *both follow MA*(K) *processes. Then, the ELM based on reported returns is lower than the true ELM if not all smoothing coefficients are equal.*

However, if market betas of the hedge funds are sufficiently different, then ELM of smoothed returns can be larger than ELM of true underlying returns. To show this possibility, consider the case in which the scale parameters γ_R, γ_{ε_1}, γ_{ε_2} all equal one and $\beta_1\theta_{1,k}<\beta_2\theta_{2,k}$ for all k. In this case, Equation (41) implies

$$\mathrm{ELM}(S_{1,t},S_{2,t})=\frac{\beta_1^{\alpha}\sum_{k=0}^{K}\theta_{1,k}^{\alpha}}{\sum_{k=0}^{K}\theta_{1,k}^{\alpha}+(1+\beta_2)\sum_{k=0}^{K}\theta_{2,k}^{\alpha}}=\frac{\beta_1^{\alpha}}{1+(1+\beta_2)\frac{\sum_{k=0}^{K}\theta_{2,k}^{\alpha}}{\sum_{k=0}^{K}\theta_{1,k}^{\alpha}}}. \quad (42)$$

Equation (42) shows that ELM depends on the ratio $\sum\theta_{2,k}^{\alpha}/\sum\theta_{1,k}^{\alpha}$. The presence of autocorrelation can either increase or decrease the estimated ELM as compared with the no-smoothing case when $\theta_{1,0}=\theta_{2,0}=1$ and $\theta_{1,k}=\theta_{2,k}=0$ for $k\neq 0$. Therefore, we have:

Proposition 8: *If the factors of the market model for two series of returns exhibit heavy tails as in Equation* (6)*, then it is not possible to establish a priori the impact of smoothing on the ELM of the two series.*

In summary, if the market betas of the hedge funds are very similar, then it can be expected that the reported returns induce a lower measure of systemic risk than is factual. However, in general, the sign of the impact of smoothing cannot be established a priori for ELM and the smoothing coefficients must be estimated to determine in which direction the reported returns bias the systemic risk measure.

2 EMPIRICAL METHODOLOGY

The theory developed in Section 1 is applied empirically using data from Hedge Fund Research (HFR). We use HFR's equally weighted total return indices denominated in U.S. dollars for the Equity Hedge and Event-Driven indices (see www.hedgefundresearch.com). Monthly returns across the period January, 1990 to August, 2013 are used, which amounts to 284 months in total.[6]

The approach described above for individuals hedge funds does not extend immediately to indices because of the issues arising from the aggregation of stochastic processes that exhibit some dependence between each other. Nevertheless, the statistical properties of hedge fund indices oftentimes are quite similar to those of individual hedge funds. In Appendix A.2, we report the results for a three-fund toy index and show that when our top-down approach is applied to this index it yields results that are similar to a bottom-up-based methodology in which the individual fund returns are unsmoothed before composing the index returns. Moreover, for indices it is easier to find longer time series and analyze the changes over time of the risk measures.

2.1 Estimation of the Smoothing Coefficients

In this subsection, the smoothing coefficients θ_k are estimated. Before doing this, we first inspect the raw returns data and their squares for their autocorrelation properties. We find that the returns of both indices exhibit significant MA(2) behavior. Following Getmansky, Lo, and Makarov (2004), we apply a maximum likelihood estimator (MLE) to the MA(K) smoothing model described in Equations (1) to (3), with $K=2$, to obtain efficient estimates of the impact of smoothing on VaR, ES, and pairwise correlation. In our numerical procedures, X_t is not required to be normally distributed, as in Getmansky, Lo, and Makarov (2004). We can still exploit the asymptotic normality of the MLE, though (cf. paragraph 8.8 in Brockwell and Davis 1991).

For ELM in the heavy-tail case a single-factor model is assumed. In this case, we obtain efficient estimates of the smoothing coefficients θ_k and the sensitivity of the hedge fund index returns to the market index returns β by applying a MLE to

[6] More details about the dataset are reported in Section 2 of Di Cesare, Stork, and de Vries (2014).

Table 1 Estimates of the smoothing coefficients.

	General framework			Market model			
Investment strategy	$\hat{\theta}_0$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_0$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\beta}$
Equity hedge	0.768	0.161	0.071	0.794	0.126	0.080	0.549
	(0.054)	(0.037)	(0.042)	(0.035)	(0.028)	(0.029)	(0.037)
Event-driven	0.689	0.248	0.063	0.710	0.216	0.074	0.430
	(0.046)	(0.029)	(0.037)	(0.030)	(0.023)	(0.026)	(0.029)
S&P 500	0.947	0.067	−0.014	1.0000	0.000	0.000	1.000
	(0.078)	(0.052)	(0.057)	(0.000)	(0.000)	(0.000)	(0.000)

Maximum likelihood estimates based on monthly log-returns for the period January 1990 to August 2013. Standard errors are reported in parentheses. The general framework refers to the MA(2) smoothing process of Equations (1) to (3). The market model refers to the linear single-factor model of Equations (1) to (4), with the S&P 500 total return index used as market factor.

the model described in Equations (1) to (4), with $K=2$. We use the S&P 500 index as the underlying factor in the estimations.

Table 1 reports the estimated smoothing coefficients for the two indices and the S&P 500 index using both the more general framework and the market model.

The results of both methods are similar. We glean from Table 1 that the estimated smoothing coefficients $\hat{\theta}_1$ and $\hat{\theta}_2$ are statistically different from zero and the parameter estimate $\hat{\theta}_0$ is statistically different from one for both hedge fund indices.

Because hedge funds change their investment exposures frequently, it is likely that the smoothing coefficients are not constant over time. As a result the calculation of the risk measures could be affected as well. For this reason we also estimate the smoothing coefficients using rolling windows of 60 months. Figures 1 and 2 report the estimates for both indices using, respectively, the more general framework and the market model. We find that the coefficients tend to remain fairly stable across the sample period 1995–2013. Furthermore, the resulting parameter levels are quite similar across the two models.

2.2 Estimation of the Tail Index and Scale Parameters

To evaluate ELM from Equations (38) and (41), in addition to the smoothing coefficients we need to estimate the tail index α and the scale parameters γ_R, γ_{ε_1}, and γ_{ε_2}. We estimate the tail index using the standard Hill (1975) estimator (see Jansen and de Vries 1991; Embrechts, Klüppelberg, and Mikosch 1997).[7] The scale parameters are estimated by approximating the probability $\mathbb{P}(X>x)\simeq\gamma x^{-\alpha}$

[7] The Hill estimator is consistent in the presence of autocorrelation in the returns or their squares; see, for example, Drees (2008).

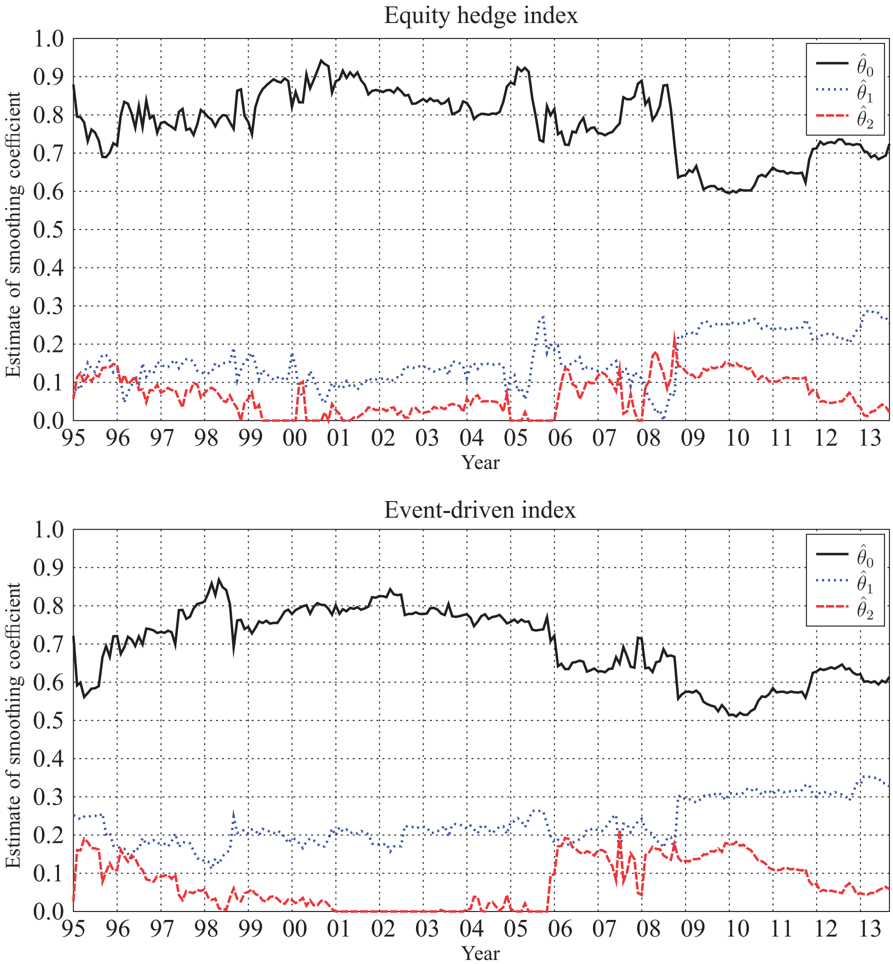

Figure 1 Maximum likelihood estimates of the smoothing coefficients (θ_k) of Equations (1) to (3). The estimations are based on rolling windows of 60 months ending in the reference month.

by its empirical value, where x is the threshold return level above which the Pareto approximation applies.

For the market factor R_t, the estimates of its tail index and its scale parameter are obtained using the S&P 500 index data. Before the tail index and the scale estimates for the residuals ε_i can be obtained, the residuals themselves need to be estimated. To this end, we first calculate

$$\hat{u}_t = S_t - \left(\hat{\mu} + \hat{\beta}\left(\hat{\theta}_0 R_t + \hat{\theta}_1 R_{t-1} + \hat{\theta}_2 R_{t-2}\right)\right), \tag{43}$$

where $\hat{\mu}, \hat{\beta}, \hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2$ are estimates obtained by the MLE described in the previous section. Then, given that $u_t = \theta_0 \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-1}$, the residuals are obtained

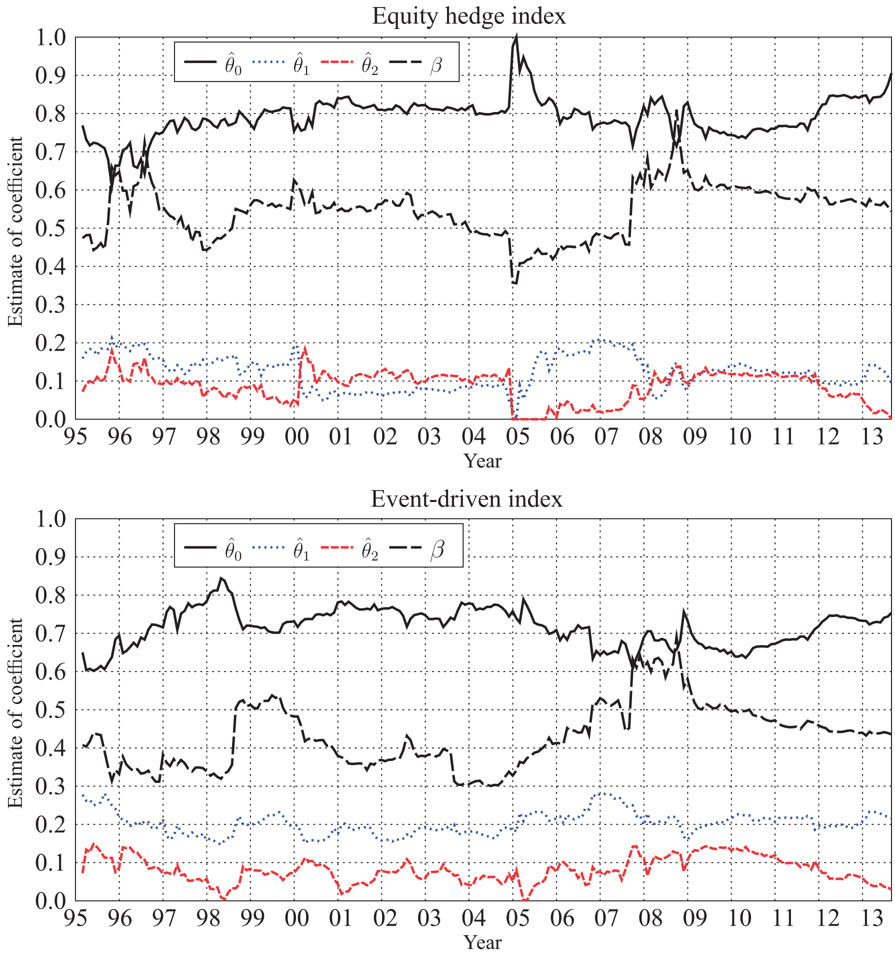

Figure 2 Maximum likelihood estimates of the smoothing coefficients (θ_k) and the market exposure (β) of the market model described by Equations (1) to (4). The estimations use the S&P 500 total return index as market factor and are based on rolling windows of 60 months ending in the reference month.

recursively using

$$\hat{\varepsilon}_t = (\hat{u}_t - \hat{\theta}_1 \hat{\varepsilon}_{t-1} - \hat{\theta}_2 \hat{\varepsilon}_{t-2}) / \hat{\theta}_0. \tag{44}$$

Given the relatively small sample size of only 284 monthly observations, we use Hill plots to determine the number of higher order statistics of the loss distributions to be used in the Hill estimator (see Embrechts, Klüppelberg, and Mikosch 1997).[8]

[8]More details about the estimation techniques and the Hill plots are reported in Section 3 of Di Cesare, Stork, and de Vries (2014).

In all cases, we find that the tail index hovers around three. This value is similar to the estimates provided by Jansen and de Vries (1991) and Hyung and de Vries (2005) for individual U.S. stocks. Given that the different estimates are close, the tail index α is set to three for both the S&P 500 index and the residuals of the hedge fund indices in the calculations that follow.

To estimate the scale parameters, we use a method analogous to the Hill plot, but with the estimates of γ on the vertical axis (fixing α at three). As the plots for the tail indices turn out to be relatively stable for the top decile of the observations, the scale parameters are set equal to the average of their estimates across those extreme returns. For the scale parameters γ_R, γ_{ε_1}, and γ_{ε_2} these mean estimates equal, respectively, 46.40, 2.99, and 1.63 .

Since the sample size of 284 monthly observations is somewhat moderate for application of extreme value methods, we investigate the relevance of small sample size and the efficiency of the estimates in Section 4 of Di Cesare, Stork, and de Vries (2014). In short, two main conclusions are drawn from a Monte Carlo simulation analysis. First, the simulations show that the estimates for VaR and ELM are essentially unbiased even for small samples. Second, the confidence intervals tend to be fairly wide in small samples but they rapidly decrease in larger samples. Hence, we conclude that our empirical estimates could be somewhat affected by the small size of the sample but any potential bias is likely to be fairly small and does not affect our main findings.

3 EFFECTS OF AUTOCORRELATION ON RISK MEASURES

In this section, we analyze how autocorrelation affects various risk measures. As hedge funds frequently modify their investment exposures, the smoothing coefficients may change over time as well. To capture these time-varying changes, rolling windows of 60 months are used in most of the following calculations.

3.1 Univariate Measures of Risk

In the case of SR, we find that the uncorrected risk measures for the equity hedge and event-driven indices equal 0.90 and 1.07, respectively (see Table 1 in Di Cesare, Stork, and de Vries 2014). After unsmoothing the returns using the correction term reported in Getmansky, Lo, and Makarov (2004, Equation 32) and our estimates of the smoothing coefficients in the general framework case (Table 1) SRs drop to 0.71 and 0.78. Although the results are not directly comparable, it is worth noting that this large change in SR exceeds the impact of unsmoothing found by Getmansky, Lo, and Makarov (2004, Table 14).[9]

[9]One difference between our empirical exercise and Getmansky, Lo, and Makarov (2004) is that we use data from HFR for the period 1990–2013 whereas the other authors use data from TASS for the period

As for VaR, Figure 3 reports the time series estimates of the reported and uncorrected VaR(S_t,p) as well as the true and unobservable VaR(X_t,p) for the two indices, calculated both under the hypothesis that returns are normally distributed (cf. Equations 12 and 13) and assuming fat-tailed returns (cf. Equations 16 and 17).

The analysis of Figure 3 yields a number of interesting conclusions. First, the uncorrected and corrected VaRs often increase at the same points in time. Apparently the dynamics of the two VaR series are fairly similar, although the corrected VaR shows larger jumps, which is an intuitive result.

Second, the levels of the two measures of VaR are substantially different. On several occasions the corrected VaR exceeds the uncorrected VaR by 50% or more. For both indices the differences between the uncorrected and corrected VaR levels increase substantially since the last months of 2008, after the collapse of Lehman Brothers. The reason for the increased discrepancy is that the effect of smoothing becomes stronger. Around this time the value of θ_0 decreases (Figure 1).

Third, VaRs are markedly higher when hedge fund returns are assumed to be fat tailed in comparison with the case of a normal distribution.

Finally, the adjustment factors in Equations (14) and (16) at the estimated parameters values are not very different. Hence, the relative difference between the corrected and uncorrected VaR series is almost the same for normal and fat-tailed returns.

In this subsection, the focus is on the VaR metric. However, for the ES similar conclusions can be drawn. The difference between the true unobservable ES and its unreported counterpart mimics that of VaR (see Equation 27). Thus, also for ES it is highly relevant to adjust the fat-tailed hedge fund returns for smoothing effects.

The previous findings illustrate the relevance of our extension of the Getmansky, Lo, and Makarov (2004) paper. Their paper shows the importance of adjusting the risk metrics of hedge fund returns for SR and beta. In those cases the underlying distributions of the returns are irrelevant, as long as the second moments exist. In practice, the returns of most assets are fat tailed (see Jansen and de Vries 1991), and we show that the Getmansky, Lo, and Makarov (2004) framework is even more pertinent for those risk measures (VaR, ES, and ELM) that depend on the specific distributions of the returns.

Finally, we note that return smoothing and autocorrelation may have other causes as well, for example, skilfull trading, option-like payoffs, database biases, return persistence, and misreporting by fund managers, as Bollen and Pool (2009) show. Not in all cases may the MA(K) representation be fully appropriate. Using simulations, we investigate how robust the MA(K) framework is in the presence of misreporting, when fund managers avoid reporting small negative returns, like in Bollen and Pool (2009). Simulations reported in Section 5 of Di Cesare, Stork, and de Vries (2014) show that also in this case VaR estimates

1977–2001. Moreover, we run our estimators on indices whereas the other authors use data on individual funds. Finally, our SR is calculated with respect to the U.S. dollar three-month Libor rate whereas the other authors use a zero interest rate benchmark.

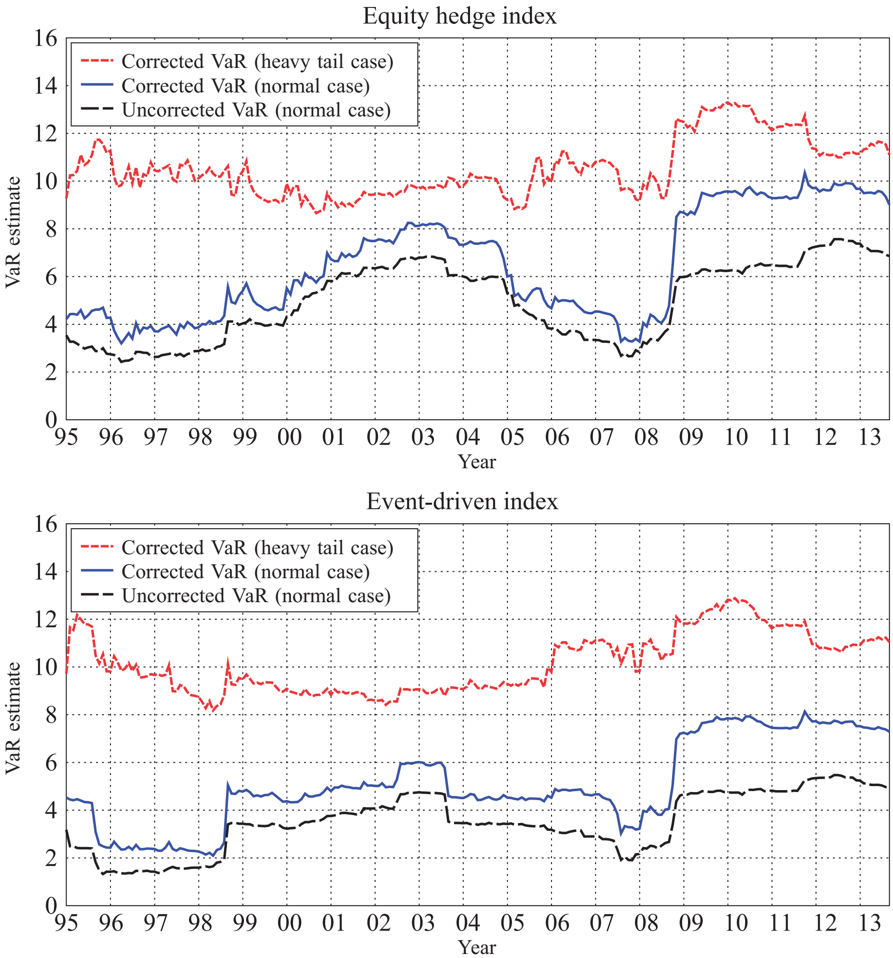

Figure 3 VaR at the 99% confidence level under the assumption of normally and fat-tailed distributed monthly returns. Percentage values. In the normal case, the uncorrected VaR (which is based on reported returns) and the corrected VaR (which is based on the true unobservable returns) are equal, respectively, to Equation (12) and Equation (13). In the heavy-tail case, the corrected VaR is equal to Equation (17); the scale factors for the reported returns are calculated over the whole sample period and are equal to 5.42 and 3.59 for the equity hedge and the event-driven index, respectively. Maximum likelihood estimates of the smoothing coefficients (θ_k) are based on Equations (1) to (3) and refer to rolling windows of 60 months ending in the reference month (cf. Figure 1).

strongly improve using the MA(K) framework of Equations (1) to (3). Introduction of misreporting around the centre of the return distribution does not affect the uncorrected VaR estimates much, and changes the outcomes to a smaller degree than the correction for smoothing itself.

Table 2 VaR as a function of the confidence level.

	Process X_t			Process S_t		
Confidence level	Model VaR	Real VaR	Difference (in percentage)	Model VaR	Real VaR	Difference (in percentage)
95.00%	2.80	2.35	19.15	2.16	1.87	15.30
97.50%	3.53	3.18	11.02	2.72	2.51	8.44
99.00%	4.80	4.54	5.62	3.70	3.55	4.12
99.90%	10.33	10.21	1.22	7.96	7.90	0.86
99.95%	13.02	12.92	0.78	10.03	9.98	0.55

The process X_t is generated by simulating 10 million i.i.d. random variables having a Student's t-distribution with three degrees of freedom. The process S_t is a moving average of X_t with smoothing coefficients equal to the general framework estimates for the equity hedge index reported in Table 1. The model VaR is calculated according to Equations (16) and (17). The real VaR is the empirical percentile.

3.1.1 Accuracy of EVT approximations.

The above results rely on EVT-based methodologies, which hold in the extreme tails. In this subsection, we analyze how well the method works for less extreme quantiles.

We provide numerical results for the case of VaR. More specifically, we run 10 million simulations of the random variable X_t, $t=1,\dots,10^7$, under the assumption that it is i.i.d. and Student's t-distributed with 3 degrees of freedom. Thus, X_t has heavy tails with tail index α equal to 3 and scale parameter A equal to $f(0;\alpha)\alpha^{(\alpha-1)/2}$, where $f(0;\alpha)$ is the density function of a Student's t-distribution with α degrees of freedom evaluated in zero (so that $A=1.1$ when $\alpha=3$). Notice that the value of the tail index corresponds to our estimate of the tail index for the two hedge fund indices used in our empirical study.

Using the 10 million observations for X_t and the general framework estimates of the smoothing coefficients for the equity hedge index reported in Table 1, we calculate the process $S_t=\sum_k \theta_k X_{t-k}$. We then compare the empirical VaR of the two processes X_t and S_t with the approximated analytical VaR (cf. Equations 16 and 17) for confidence levels between 95.00% and 99.95%. Given the high number of simulations, we are confident that the empirical VaR is indeed close to the true unobservable VaR of the two processes.

Table 2 shows that the model VaR is quite close to the real VaR for confidence levels often used in practice (99% and higher), thus providing support to the accuracy of our methodology for higher ranging quantiles. However, as expected the application of EVT techniques to lower quantiles is less effective. Results obtained using the estimated smoothing coefficients for the event-driven index are qualitatively similar and are reported in Table 6 of Di Cesare, Stork, and de Vries (2014).

3.2 Bivariate Measures of Risk

Next, we discuss how autocorrelation impacts on the bivariate measures of risk. First, consider the correlation measure. Using the estimates reported in Figure 1,

Figure 4 ELM between the equity hedge and event-driven indices. The uncorrected ELM (which is based on reported returns) and the corrected ELM (which is based on the true unobservable returns) are equal, respectively, to Equation (38) and Equation (41). The smoothing coefficients (θ_k) and the market exposure (β) of the market model described by Equations (1) to (4) are estimated by MLE; the estimations use the S&P 500 total return index as market factor and are based on rolling windows of 60 months ending in the reference month (cf. Figure 2). The tail index (α) and the scale parameter of the market factor (γ_R), estimated using Hill plots, are kept fixed at 3 and 46.40, respectively. The scale parameters (γ_ε) of the idiosyncratic terms of the two indices, estimated using Hill plots, are kept fixed at 2.99and 1.63 , respectively.

the correction term in Equation (28) ranges between 0.95 and 1.00, so that the impact of smoothing on correlation is almost negligible.[10] Note that this correction term can be interpreted as the raw correlation between the smoothing coefficients. Thus, the similarity in the autocorrelation structure of the equity hedge and event-driven indices implies that the estimated correlation is hardly affected by smoothing.

Given the presence of heavy tails, a better measure for the tail dependence is ELM. Using maximum likelihood estimates over the entire sample period for the smoothing coefficients θ_k and the exposure to the market factor β from Table 1 and the scale parameters γ reported above, we calculate the two ELM measures according to Equations (38) and (41) and find that $\text{ELM}(X_{1,t}, X_{2,t}) = 0.30$ and $\text{ELM}(S_{1,t}, S_{2,t}) = 0.22$. Thus, the true measure of systemic risk is ~35% higher than ELM based on smoothed returns (see Section 4 of Di Cesare, Stork, and de Vries (2014) for some evidence about the estimation errors in these estimates).

Figure 4 depicts ELM estimates for the two indices based on rolling windows of 60 months. It shows that most of the time the smoothing-adjusted ELM significantly exceeds its unadjusted counterpart, confirming the estimates based on the whole sample. At times, the relative size of the adjustment strongly exceeds the 50% estimation error that we found above. For instance, at the end of the sample period

[10]See Figure 6 in Di Cesare, Stork, and de Vries (2014) for the estimated time series of the correlation.

the corrected probability of one hedge fund index being under stress given that the other index is under stress is ~65% higher than the uncorrected probability.

Evidently, unsmoothing the observed hedge fund returns is especially important when studying the extreme tail dependence. The large differences in the behavior of the correlation and ELM estimates underscores the relevance of our proposed adjustments.

4 CONCLUSION

Hedge fund returns frequently exhibit a strong degree of autocorrelation. As a result, the economic risks of an investment in hedge funds are easily underestimated and investment decisions can become biased. In this article, we extend the seminal work of Getmansky, Lo, and Makarov (2004) on SR and market beta, by developing a number of smoothing-adjusted downside risk measures and by allowing for non normal fat-tailed return distributions. In particular, both individual risk measures (VaR and ES) and systemic risk measures (pairwise correlation and ELM) are adjusted for the autocorrelation present in reported returns. We show that the adjustment of the downside risk measure ELM for autocorrelation is more important when returns are fat tailed than when they are normally distributed. A hedge fund index case study reveals that unadjusted risk measures can considerably underestimate the true extent of individual and multivariate risks. Finally, we note that, although our risk-adjustment is applied to hedge funds only, our framework can also be used to evaluate the risks of other alternative investment strategies. Investments in real estate, art, collectible stamps, and other illiquid or opaque securities are also known to exhibit strong serial correlation in the reported returns. For these assets, conventional risk measures also need adjustments to correctly reflect the true level of investment risk.

APPENDIX A

A.1 Proof of Proposition 7

Given the equal betas, the true $\text{ELM}(X_{1,t},X_{2,t})$ from Equation (38) reduces to

$$\text{ELM}(X_{1,t},X_{2,t}) = \frac{\beta^\alpha \gamma_R}{\gamma_{\varepsilon_1}+\gamma_{\varepsilon_2}+\beta^\alpha\gamma_R} = \frac{1}{\frac{\gamma_{\varepsilon_1}+\gamma_{\varepsilon_2}}{\beta^\alpha\gamma_R}+1}. \tag{A.1}$$

The corresponding measure for the smoothed returns from Equation (41) becomes

$$\text{ELM}(S_{1,t},S_{2,t}) = \frac{\sum_{k=0}^{K}\left(\min(\theta_{1,k},\theta_{2,k})\right)^\alpha}{\frac{\gamma_{\varepsilon_1}}{\beta^\alpha\gamma_R}\sum_{k=0}^{K}\theta_{1,k}^\alpha+\frac{\gamma_{\varepsilon_2}}{\beta^\alpha\gamma_R}\sum_{k=0}^{K}\theta_{2,k}^\alpha+\sum_{k=0}^{K}\left(\max(\theta_{1,k},\theta_{2,k})\right)^\alpha}. \tag{A.2}$$

Comparing the two measures $\mathrm{ELM}(X_{1,t}, X_{2,t}) \gtreqless \mathrm{ELM}(S_{1,t}, S_{2,t})$ shows

$$\begin{aligned} &\frac{\gamma_{\varepsilon_1}}{\beta^\alpha \gamma_R} \sum_{k=0}^{K} \theta_{1,k}^\alpha + \frac{\gamma_{\varepsilon_2}}{\beta^\alpha \gamma_R} \sum_{k=0}^{K} \theta_{2,k}^\alpha + \sum_{k=0}^{K} \left(\max(\theta_{1,k}, \theta_{2,k})\right)^\alpha \gtreqless \\ &\qquad \frac{\gamma_{\varepsilon_1} + \gamma_{\varepsilon_2}}{\beta^\alpha \gamma_R} \sum_{k=0}^{K} \left(\min(\theta_{1,k}, \theta_{2,k})\right)^\alpha + \sum_{k=0}^{K} \left(\min(\theta_{1,k}, \theta_{2,k})\right)^\alpha, \end{aligned} \tag{A.3}$$

or

$$\begin{aligned} &\sum_{k=0}^{K} \left(\left(\max(\theta_{1,k}, \theta_{2,k})\right)^\alpha - \left(\min(\theta_{1,k}, \theta_{2,k})\right)^\alpha \right) \gtreqless \\ &\frac{\gamma_{\varepsilon_1}}{\beta^\alpha \gamma_R} \sum_{k=0}^{K} \left(\left(\min(\theta_{1,k}, \theta_{2,k})\right)^\alpha - \theta_{1,k}^\alpha \right) + \frac{\gamma_{\varepsilon_2}}{\beta^\alpha \gamma_R} \sum_{k=0}^{K} \left(\left(\min(\theta_{1,k}, \theta_{2,k})\right)^\alpha - \theta_{2,k}^\alpha \right). \end{aligned} \tag{A.4}$$

The elements on the left-hand side are all nonnegative (and some are strictly positive if not all smoothing coefficients are equal), while the terms on the right-hand side are all nonpositive (and some are strictly negative if not all smoothing coefficients are equal). Hence, the left-hand side is always at least as large as the right-hand side, or $\mathrm{ELM}(X_{1,t}, X_{2,t}) \geq \mathrm{ELM}(S_{1,t}, S_{2,t})$. ■

A.2 Application to Indices

It is straightforward to show that a linear combination (an index) of individual hedge fund returns that behave according to Equations (1) to (3) does not necessarily satisfy the same equations, unless all hedge funds have the same smoothing coefficients. In addition, hedge fund returns may show some interdependence, as in Equation (4), so that Feller's theorem cannot be used directly to study the stochastic properties of their linear combinations.

Thus, to unsmooth index returns properly one needs to use the properties of individual hedge fund returns and their interdependencies. Here, we compare the VaR of an index made of three hedge funds according to two methodologies. First, we use the top-down approach developed in the article, that is we deal with the index as if it were an individual hedge fund. In the second approach, we estimate the VaR of the unsmoothed index using the individual fund characteristics and their interdependence.

The three hedge funds are among the largest funds (in terms of assets under management) with data available on Bloomberg, and with a relatively long time series of data (Table A.1). We have 209 monthly observations from January 1996 to May 2013. The "General framework" part of Table A.2 shows that the returns are MA(2) for two funds and MA(1) for the third. Index returns display MA(1) behavior.

Table A.1 Main characteristics of the funds used in this Appendix.

Name	Bloomberg ID	Assets ($ bn)	Strategy
Egerton European dollar	EGNEDFI VI	3.3	Long short
Odey European	ODYEDMI KY	1.6	Macro discretionary
LIM Asia multi-strategy	LIMASFI VI	1.2	Macro discretionary

Table A.2 Estimates of the smoothing coefficients.

	General framework			Market model			
Investment strategy	$\hat{\theta}_0$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_0$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\beta}$
Fund 1	0.744	0.103	0.153	0.785	0.081	0.134	0.361
	(0.056)	(0.045)	(0.043)	(0.055)	(0.044)	(0.045)	(0.052)
Fund 2	0.842	0.159	0.000	0.850	0.141	0.009	0.444
	(0.076)	(0.048)	(0.058)	(0.064)	(0.046)	(0.047)	(0.073)
Fund 3	0.645	0.225	0.130	0.671	0.214	0.115	0.163
	(0.045)	(0.032)	(0.037)	(0.053)	(0.033)	(0.039)	(0.035)
Index	0.771	0.162	0.067	0.797	0.136	0.068	0.326
	(0.063)	(0.044)	(0.049)	(0.055)	(0.040)	(0.042)	(0.041)

Maximum likelihood estimates based on monthly log-returns for the period January 1996 to May 2013. Standard errors are reported in parentheses. The general framework refers to the MA(2) smoothing process of Equations (1) to (3). The market model refers to the linear single-factor model of Equations (1) to (4), with the S&P 500 total return index used as market factor. Monthly returns of the index are calculated as simple averages of individual funds monthly returns.

The analysis of the Hill plots shows that a tail index equal to three is a good estimate, both for the individual funds and the index. The scale parameters, estimated as described in Section 2.2, are equal to 8.21, 16.55, and 1.27 for the three funds and 4.03 for the index. Using these results, the VaR of the index at the 99% confidence level is equal to 7.38% if reported returns are used (see Equation 16) and to 9.55% if returns are corrected for smoothing (see Equation 17). The corrected VaR is thus almost 30% higher that the uncorrected measure.

We then assume that individual hedge fund returns behave according to Equations (1) to (4), so that they also show some dependence derived from the common risk factor (which is assumed to be the S&P 500). The estimated smoothing coefficients and market exposure are reported in the "Market model" part of Table A.2. Using these parameters, we estimate the residuals of fund returns (see Equation 44) and their scale parameters, which are equal to 12.04, 26.15, and 3.74, respectively. The scale parameter for the S&P 500 is 59.21.

For an equally weighted index of N hedge funds with returns that behave according to Equations (1) to (4), the correction term VaRCT to unsmooth the index returns is

$$\text{VaRCT}=\left(\frac{\gamma_R\left(\sum_{n=1}^{N}\beta_n\right)^{\alpha}+\sum_{n=1}^{N}\gamma_{\varepsilon_n}}{\sum_{k=0}^{K}\left[\gamma_R\left(\sum_{n=1}^{N}\beta_n\theta_{n,k}\right)^{\alpha}+\sum_{n=1}^{N}\gamma_{\varepsilon_n}\theta_{n,k}^{\alpha}\right]}\right)^{\frac{1}{\alpha}}. \tag{A.5}$$

Using our estimates, the VaRCT is 1.24. By applying this VaRCT to the VaR calculated on reported returns when the index is treated as if it were an individual fund (7.38%), we obtain an estimate for the corrected VaR equal to 9.14%. This is only ∼4% lower than what we estimated previously (9.55%) using the top-down approach.

Hence, estimating the correct VaR for an index by treating the index itself as an individual fund does not seem to deliver results that are significantly different from those obtained using the more elaborate procedure based on the estimation of the parameters of each individual fund and the subsequent aggregation of the results.

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