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Simulating and calibrating diversification against black swans

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ABSTRACT

An investor concerned with the downside risk of a black swan only needs a small portfolio to reap the benefits from diversification. This matches actual portfolio sizes, but does contrast with received wisdom from mean–variance analysis and intuition regarding fat tailed distributed returns. The concern for downside risk and the fat tail property of the distribution of returns can explain the low portfolio diversification. A simulation and calibration study is used to demonstrate the relevance of the theory and to disentangle the relative importance of the different effects.

1. Introduction

Early diversification studies, such as that of Evans and Archer (1968) and Elton and Gruber (1977) have focussed on the benefits from portfolio diversification that can be measured by the volatility of portfolio returns. These studies consider how many randomly selected stocks to include in a portfolio before most idiosyncratic risk is eliminated. Fama (1976), for example, identified the portfolio size that generates a 95% reduction in portfolio variance. Evans and Archer (1968) advocated a statistical criterion to determine the optimal portfolio size using the number of stocks at the point where no further significant reduction in the portfolio dispersion can be obtained. More recently, Campbell et al. (2001, p. 25) concluded that in more recent decades: ‘the increase in idiosyncratic volatility over time has increased the number of randomly selected stocks needed to achieve relatively complete portfolio diversification’.[1]

Although it is clearly worthwhile and interesting to determine how much it would take to eliminate ‘almost all’ unsystematic risk through diversification, it is unsatisfactory in an economic sense when left to itself. An economic based portfolio size not only assesses the benefits in terms of risk reduction, but also takes into account the associated costs of diversification. Diversification should be increased as long as the marginal benefits exceed the marginal costs of adding

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[1] See also Tang (2004), Domian et al. (2007) and Benjelloun (2010).

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one extra security. Statman (1987) has contributed to the literature on diversification by proposing a framework in which the costs and benefits can be balanced. However, even if the costs of diversification are adequately assessed, Statman (1987, 2004) nevertheless conclude that the level of diversification in investor's equity portfolios presents a puzzle regarding the mean–variance based portfolio analysis. The level of diversification in the average investor's portfolio found empirically, which hovers at around three or four stocks, appears to be far less than the optimal theoretical level. Consider, though, the *Financial Times* of August 1, 2011 weekly review of fund management, which opened with the headline: 'New research shows concentrated funds outperform diversified vehicles'. What should be made of this?

In this paper we investigate the benefits of diversification for an investor who has a concern over downside risk and recognizes the fat tail feature in the distribution of asset returns, while retaining concern regarding the costs of diversification. The concern for downside risk is captured by the value-at-risk (VaR), which has become a popular measure of risk in the banking industry. One might think that fatter than normal tails, that give rise to more outliers, would require even more diversification than in the case of the standard assumption of normally distributed returns. We show, though, that at a given level of (downside) risk, the benefits from diversification come in more rapidly in the case of fat tails. The apparent contradiction stems from the fact that in comparing the tails, one is comparing distributions as the risk level changes. But diversification stems from cross-sectional aggregation at a constant risk level. Diversification at a constant risk level reduces the VaR, but this occurs at a rate that is not necessarily equal to the rate if the risk level is reduced. As long as the second moment is bounded, our simulations and derivations show that the Rate of Diversification for fat tailed distributed returns in terms of reducing the VaR can be better than for normally distributed returns. Ibragimov (2009) considers the case of infinite variance, wherein the effects go in the opposite direction (and diversification may not be a good idea at all).

The downside risk concern is modeled within the safety first framework of Roy (1952) through a VaR constraint.[2] Arzac and Bawa (1977) provide an equilibrium analysis, like the CAPM, of the safety-first investor who maximizes expected returns subject to a VaR constraint. Downside risk and the closely related safety-first principle are frequently employed in models of finance and risk management, providing alternatives to the traditional expected utility framework. More recently, the concern for downside risk has received renewed interest, see e.g. Gourieroux et al. (2000), Jansen et al. (2000), and Campbell and Kraussl (2007).

To assess the diversification issue, we consider equally weighted randomly composed portfolios. Kan and Zhou (2007) discuss the difficulties in estimating the parameters, such as the mean and variance, required to construct the optimal portfolio weights. Dash and Loggie (2008) suggest that the equally weighted index turns out to be a powerful investment idea after examining the performance of the S&P 500 Equal Weight Index. Equally weighted portfolios circumvent the difficulty of having to estimate optimal portfolio weights. Moreover, this facilitates an easier comparison across different risk measures, as different utility functions imply different optimal portfolio weights.

To compare the incremental benefits and costs from diversification for different risk measures, define the 'Rate of Diversification' as the derivative of the benefits with regard to the number of assets. For independent risk drivers that are either normally or fat tailed distributed, it is relatively straightforward to obtain the rates of diversification. In an equilibrium setting, when stock returns are dependent, there can be multiple rates stemming from different contributing factors. Since it can not be determined *a priori* which source dominates with a limited number of assets, we employ a numerical calibration and simulation study. These calibrations and simulations complement the empirical work in our companion paper, see Hyung and de Vries (2010).

We find that, in comparison to a mean–variance investor, the concern for extreme[3] downside risk in combination with the fat tail feature produces more focused portfolios. At first, this may seem counter-intuitive. However, as we show, the normal distribution is conducive to high diversification, as this reduces the power by which the (exponential type) tail of the loss distribution declines. Per contrast, diversification only affects the scale of fat tailed distributions, which is a more limited effect.

In Section 2 we briefly review the analysis of Statman (2004) and Hyung and de Vries (2010). Section 3 presents analytical expressions for the rates of the diversification benefits. Following this, we conduct a calibration and simulation study in Sections 4 and 5, respectively. Conclusions are provided in the final section.

2. Risk measures and diversification cost benefit analysis

The costs of diversification are the transaction costs in buying and selling, holding costs and monitoring costs of assets. Statman (2004) defines the concept of 'additional net cost' as the net cost of increasing diversification from any n-stock portfolio to a fully diversified portfolio. In accordance with the work of Statman, we assume constant net additional costs. The optimal portfolio size is the point at which the marginal cost of adding one extra security equals the benefit of the reduction in risk. Adding the cost side reduces the amount of diversification.

[2] We have also considered the expected-shortfall constraint, which provides similar results. For brevity of presentation we do not present these results; however, they are available on request from the authors.

[3] The terminology of 'extreme risk' refers to risk levels such as 0.1% at daily frequency. This corresponds to approximately one event per five years. The non-extreme risk levels are the 5% or 1% levels used commonly in VaR exercises by banks.

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The benefit of diversification is the reduction of risk. Since the diversification costs are expressed in currency units, the benefits have to be brought under the same numeraire in order to be able to determine the optimal level of diversification. To this end, the benefits are translated into units of expected returns. In order to accomplish this, the risk reduction benefits of diversification in units of expected return are determined by a simple comparison of two portfolios along the 'Total Market Line' as elucidated by Statman (2004).

2.1. Mean-variance diversification

Let the m-stock portfolio, $P(m)$, denote the market or tangency portfolio from the two fund separation result. The market portfolio has expected return $E[r(m)]=R$ and standard deviation σ_m. From the perspective of the uninformed investor or random stock picker, all stocks and hence portfolios are assumed to have the same expected returns as the market portfolio, R. Moreover, individual stocks are assumed to have identical *a priori* risk characteristics (variance, correlation, VaR). The return R equals the sum of the risk-free rate, R_f, plus the equity premium (denoted as EP), i.e. $EP=R-R_f$.

Let $P(n)$ denote a portfolio with size n, $n<m$, with standard deviation σ_n, $\sigma_n>\sigma_m$, and having expected return $E[r(n)]=R$ by assumption. If investors can borrow and lend at the risk-free rate, the m-stock portfolio can be levered through borrowing to form the levered portfolio $P(n^*)$ that lies higher up on the market line. The leverage linearly alters the standard deviation σ_m of the market portfolio in accordance with the market line, say, to σ_n. The standard deviation of the levered portfolio $P(n^*)$ thus equals the standard deviation of the less diversified (unlevered) n-stock portfolio, but has a higher expected return as it is located on the market line

$$R_{n^*}=R_f+\frac{\sigma_n}{\sigma_m}EP, \tag{1}$$

and where R_{n^*} is the expected return of the levered portfolio $P(n^*)$.

Eq. (1) defines the 'Total (capital) Market Line' and all the levered portfolios $P(n^*)$ are located on this line, see Fig. 1. Consider the difference in expected returns between the n-stock portfolio $P(n)$ and the levered m-stock portfolio $P(n^*)$. This is the difference between R and the expected return R_{n^*} indicated by the market line. The incremental benefit of increased diversification from n to m stocks can then be measured in the money metric as the difference $B_n=R_{n^*}-R$. Note that using (1)

$$B_n=R_{n^*}-R=\left(\frac{\sigma_n}{\sigma_m}-1\right)\times EP. \tag{2}$$

Since the benefit in this framework derives from the change in the standard deviation, we refer to the benefit as B_n^{stdv}.

2.2. Downside risk measure

Investors considering the tradeoff between the mean and variance employ a global measure of risk. At the time Markowitz conceived the mean–variance portfolio selection theory, Roy (1952) had already proposed the alternative safety-first theory with a concern for downside risk. Interest in the safety-first criterion was rekindled by the practice in the financial industry to employ the VaR concept. Behavioral finance has given this interest a further boost.

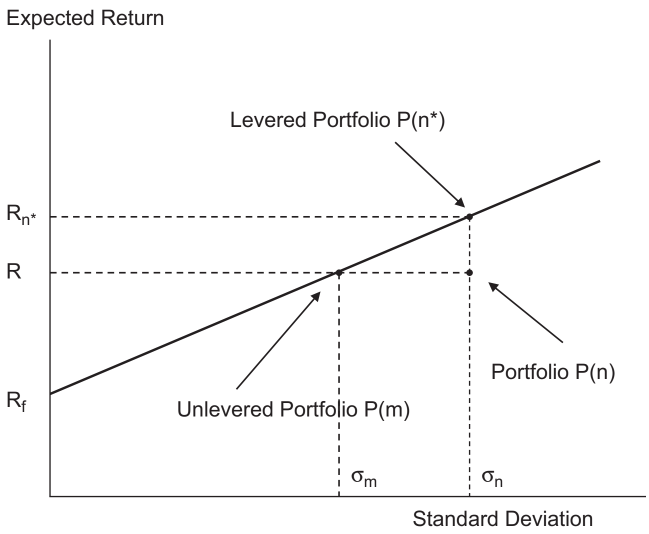

Fig. 1. Total Market Line in mean–variance theory (*Source*: Statman, 2004).

The value-at-risk (VaR) is the most popular downside risk measure in the practice of risk management. The VaR is simply a low probability high loss quantile. The VaR q at some desired probability level δ is defined as follows:

$$\Pr\{x \le -q\} = \delta.$$

Consider the portfolio choice of a safety-first investor. From the equilibrium analysis of the safety-first investor in Arzac and Bawa (1977), one obtains a relation similar to (1) in the mean-VaR context. If the m-stock portfolio $P(m)$ is levered with the risk-free asset with weight ω, we obtain the levered portfolio $P(n^*)$ with the expected return of $R_{n^*} = \omega R + (1-\omega)R_f$ located on the equivalent of the market line. The VaR of the levered portfolio then as follows:

$$\mathrm{VaR}_{n^*} = \omega\, \mathrm{VaR}_m - (1-\omega)R_f, \tag{3}$$

in which VaR_i is the value-at-risk of portfolio $P(i)$ at a given δ level. Take $P(n^*)$ such that the VaR_{n^*} is equal to VaR_n, the VaR of a less well diversified n-stock portfolio $P(n)$, $n < m$.

The expected return of the levered portfolio can thus be expressed as

$$R_{n^*} = R_f + \frac{\mathrm{VaR}_n + R_f}{\mathrm{VaR}_m + R_f} EP, \tag{4}$$

by substituting ω from (3) into $R_{n^*} = R_f + \omega(R - R_f)$. Note that this equation corresponds to Eq. (14) from Arzac and Bawa's (1977) equilibrium analysis. The incremental benefit of increased diversification from n to m stocks on the basis of the VaR measure thus reads as follows:

$$B_n^{VaR} = \left(\frac{\mathrm{VaR}_n + R_f}{\mathrm{VaR}_m + R_f} - 1 \right) \times EP, \tag{5}$$

where the superindex in B_n^{VaR} refers to the downside risk measure. In conclusion, it is fairly straightforward to adapt Statman's incremental benefit of diversification measure (2) to the case of the VaR downside risk measure, as is shown in (5).

3. Normal versus fat-tailed distribution

We compare the benefits from diversification for different types of investors who alternatively employ the VaR and standard deviation as measures of risk. The two measures imply quite different levels of diversification depending on whether the returns follow a normal distribution or a distribution that exhibits fat tails.

3.1. Diversification in case of heavy tails

In this subsection, the diversification benefits on the basis of the VaR criterion are compared with the benefits derived from the standard deviation criterion, under the assumption that the returns are heavy tail distributed. We demonstrate that diversification under the mean-VaR criterion implies smaller portfolios than under the mean–variance criterion. First, we consider the return of securities that are identically and independently distributed with heavy tails in the sense of regular variation at infinity. In the next section, this counterfactual assumption is relaxed by allowing for common factors and heterogeneous scales.

The fat tail property is modeled by assuming that the distribution in the tail areas behaves like a Pareto distribution; see Jansen and de Vries (1991) for the empirical relevance. The tail of the Pareto distribution declines at a power rate. In comparison to the normal distribution, the distribution of asset returns has more returns concentrated in the very center and more returns in the tails of the distribution. Far from the origin, the Pareto term dominates

$$\Pr\{r_i \le -s\} = As^{-\alpha}[1+o(1)], \tag{6}$$

as $\to\infty$, where $\alpha > 0$, $A > 0$ and r_i denote the return of security i. This tail relation defines an entire class of distributions such as the Student's t, non-normal sum stable and the Frechet distributions. For symmetric distributions such as the Student distribution, it holds that for large s

$$\Pr\{r_i > s\} = \Pr\{r_i \le -s\} = As^{-\alpha}[1+o(1)].$$

There is evidence, however, that asset return distributions are skewed and exhibit downside tails that are fatter than the upside tail. In the following, we only require (6) as we focus on the downside risk and we can allow for distributions that are not symmetric and may have thinner upper tails.

The class (6) satisfies a self-similarity property in the tail area. The Feller convolution theorem (1971, VIII.8) for the sum of random variables satisfying (6) holds that one can sum the tail probabilities at a large quantile s. That is, for r_1 and r_2 independently distributed and satisfying (6), for the left tails

$$\Pr\{r_1 + r_2 \le -s\} = 2As^{-\alpha}[1+o(1)],$$

as $s \to \infty$. This additivity property gives these distributions the self scaling property in the tail areas first noted in economics by Mandelbrot. The average return for an equally weighted portfolio then follows readily as

$$\Pr\{\tfrac{1}{2}(r_1 + r_2) \le -s\} = \Pr\{r_1 + r_2 \le -2s\} = 2A(2s)^{-\alpha}[1+o(1)].$$

Hence, for the return of an equally weighted n-stock portfolio $r(n) = (1/n)\sum_{i=1}^{n} r_i$, the probability $\Pr\{r(n) \leq -s\}$ for s large reads

$$\Pr\{r(n) \leq -s\} \simeq n^{1-\alpha} A s^{-\alpha}[1+o(1)]. \tag{7}$$

Next, hold the probability level δ constant but let the VaR level adapt as the number of assets n increases in $\Pr\{r(n) \leq -\mathrm{VaR}_n\} = \delta$. By first order inversion, one obtains the following: As $\delta \to 0$,

$$\mathrm{VaR}_n = (A/\delta)^{1/\alpha} n^{1/\alpha - 1}[1+o(1)]. \tag{8}$$

This result can now be used in (5) to obtain an explicit expression for the benefit of diversification for the mean-VaR safety-first investors when asset returns are heavy tailed distributed and a low acceptable risk level δ. Specifically define the benefits again as $B_n \equiv R_{n*} - R$. Then combining (5) and (8) gives (recall $n < m$)

$$B_n^{\mathrm{VaR}} = \left\{ \frac{(A/\delta)^{1/\alpha} n^{1/\alpha-1}[1+o(1)] + R_f}{(A/\delta)^{1/\alpha} m^{1/\alpha-1}[1+o(1)] + R_f} - 1 \right\} \times EP. \tag{9}$$

Before we proceed to the mean–variance criterion, it is worth pointing out that the i.i.d. assumption concerning the r_i that satisfies (6) translate the equally weighted portfolio into the global minimum VaR portfolio after solving for the optimal weights in the mean-VaR portfolio framework. This stems from the convexity of the VaR measure if $\alpha > 1$.

In order to gauge the optimal level of diversification for mean–variance investors, we begin with the equation for the standard deviation of the n stock portfolio. From the perspective of a random stockpicker, the returns r_i of different assets i are identically and independently distributed with mean R and variance σ^2 for all stocks (implicitly this requires $\alpha > 2$ in (6) to guarantee a finite second moment). Thus, we initially assume zero cross correlation $\mathrm{Cov}(r_i, r_j) = 0$, for $i \neq j$. Consider an investor who composes an equally weighted n-stock portfolio by randomly selecting n different securities from this universe of m securities. The standard deviation of an equally weighted portfolio of n stocks would then read

$$\sigma_n = \sigma/\sqrt{n}.$$

Evidently, the standard deviation of the portfolio declines as the number of stocks in the portfolio increases. The benefit of increased diversification from n to m stocks (recall $n < m$) in the specific case of (2) can therefore be expressed as

$$B_n^{stdv} = \left(\sqrt{\frac{m}{n}} - 1 \right) \times EP. \tag{10}$$

From (9) and (10) we see that for very large values of n

$$B_n^{stdv} \simeq 0 \quad \text{and} \quad B_n^{VaR} \simeq 0.$$

However, the rates at which the benefits decline towards this limit value vary. We study these rates more explicitly.

Define the instantaneous ‘Rate of Diversification’, abbreviated as RD, as the derivative of the excess benefit with regard to the number of assets n. Write

$$B_n^{stdv} = c_1 n^{-1/2} - c_2,$$

where $c_1 = m^{1/2} EP$ and $c_2 = EP$. Hence, the RD of B_n^{stdv} is

$$\mathrm{RD}\{B_n^{stdv}\} = -\tfrac{1}{2} c_1 n^{-3/2}. \tag{11}$$

From (10) one sees that $B_n^{stdv} \to 0$ as n increases, which is equivalent to the rate by which $\sigma_n \to \sigma_m$ as n increases. Note that the factor $-1/2$ in this expression signifies the square root rule $1/\sqrt{n}$.

Analogously, from (9)

$$B_n^{VaR} \simeq c_1' n^{-1+1/\alpha} - c_2',$$

where

$$c_1' = \frac{(A/\delta)^{1/\alpha} EP}{(A/\delta)^{1/\alpha} m^{1/\alpha-1} + R_f}, \quad c_2' = \frac{(A/\delta)^{1/\alpha} m^{1/\alpha-1} EP}{(A/\delta)^{1/\alpha} m^{1/\alpha-1} + R_f}.$$

Taking the derivative gives the RD of B_n^{VaR}

$$\mathrm{RD}\{B_n^{VaR}\} = -\left(1 - \frac{1}{\alpha}\right) c_1' n^{-2+1/\alpha}. \tag{12}$$

We can now obtain an important conclusion by comparing the two rates (11) and (12). Assume that asset returns are heavy tailed distributed. Provided that $\alpha > 2$, implying that the variance is finite, the diversification rate of the downside risk criterion VaR dominates the standard deviation based criterion at larger values of n, since

$$\frac{\mathrm{RD}\{B_n^{VaR}\}}{\mathrm{RD}\{B_n^{stdv}\}} = 2\left(1 - \frac{1}{\alpha}\right) \frac{c_1'}{c_1} n^{-1/2+1/\alpha} \tag{13}$$

and

$$\lim_{n\to\infty} n^{1/\alpha-1/2}=0.$$

Thus, in the case of heavy tail distributed returns, the diversification benefits for investors concerned with downside risk come in more rapidly than for a mean–variance type investor. This may appear counter-intuitive, since the heavy tail feature implies higher tail risk at VaR levels deep enough into the tail (low δ levels) in comparison to the risk under the normal law; cf. the normal distribution case considered in the subsection below. But in comparing the effects of diversification, one does not go deeper into the tail areas as the probability level δ is held constant. Rather, one studies the cross-sectional aggregation (convolution) effect. For the normal model, diversification reduces the variance and thereby increases the rate by which the tail risk declines by the square root of the number of assets (through the power in the exponent[4]). By way of contrast, diversification only affects the scale of heavy tailed distributed returns, not the power, see the convolution (7). The rate effect of the normal distribution dominates over the scale effect for the heavy tailed distributions, in the sense that it works too well and one likes to continue adding other assets.

3.2. Factor models with heavy tails

We now relax the counterfactual assumption that asset returns are i.i.d. by allowing for common factors and heterogeneous scales. This covers the standard CAPM and linear factor models such as APT. Consider the single factor model defined by the following relationship between an asset return r_i, the return r_{mkt} on the market portfolio and the idiosyncratic noise q_i

$$r_i=\beta_i r_{mkt}+q_i, \tag{14}$$

where β_i signifies the amount of market risk. The idiosyncratic risk may be diversified away in large portfolios, but the market risk reduces the benefits of diversification as it poses non-diversifiable risk.

The return of an equally weighted n-stock portfolio follows:

$$r(n)=\overline{\beta} r_{mkt}+\frac{1}{n}\sum_{i=1}^{n} q_i,$$

where $\overline{\beta}=(1/n)\sum_{i=1}^{n}\beta_i$. Assume that the market risk r_{mkt} is distributed with mean R and variance σ^2_{mkt} and the beta of stock i is also a random variable β_i with mean β and variance σ^2_β. As in the previous section we assume that the idiosyncratic risk q_i is distributed independently but not necessarily identically with mean 0 and variance $\sigma^2_{q_i}$ for all i. We further assume that the idiosyncratic risks are independent, i.e. have zero cross correlation $\text{Cov}(q_i,q_j)=0$ for $i\neq j$. Note that due to the random stock selection the stockpicker also views the idiosyncratic risks as exhibiting equal variance $\overline{\sigma}^2_q=(1/m)\sum_{i=1}^{m}\sigma^2_{q_i}$, even though the variances of the various q_i may differ.

To determine the optimal level of diversification, we begin with the standard deviation of an n stock portfolio to determine the benefits of diversification for the mean–variance optimizing investor. Consider again an investor who composes an equally weighted n-stock portfolio by randomly selecting n different securities from the universe of m securities, $n<m$. The expected standard deviation of the equally weighted portfolio of n stocks then reads as follows

$$\sigma_n=\sqrt{\beta^2\sigma^2_{mkt}+\frac{1}{n}(\sigma^2_{mkt}\sigma^2_\beta+R^2\sigma^2_\beta+\overline{\sigma}^2_q)}. \tag{15}$$

The equivalent of (10) becomes

$$B_n^{stdv}=\left(\sqrt{\frac{\beta^2\sigma^2_{mkt}+\frac{1}{n}(\sigma^2_{mkt}\sigma^2_\beta+R^2\sigma^2_\beta+\overline{\sigma}^2_q)}{\beta^2\sigma^2_{mkt}+\frac{1}{m}(\sigma^2_{mkt}\sigma^2_\beta+R^2\sigma^2_\beta+\overline{\sigma}^2_q)}}-1\right)\times EP. \tag{16}$$

Note that without a market factor, i.e. if $\beta=0$, (16) reduces to (10). The RD of B_n^{stdv} follows from differentiating (16) with respect to n

$$\text{RD}\{B_n^{stdv}\}=-\frac{1}{2}\frac{EP}{\sigma_m\sigma_n}(\sigma^2_{mkt}\sigma^2_\beta+R^2\sigma^2_\beta+\overline{\sigma}^2_q)n^{-2}. \tag{17}$$

Next we turn to the investors with a concern for downside risk. Suppose that the distributions of q_i and r_{mkt} in the single index model (14) are regularly varying with the same tail index α, but have different scales A_i and C, respectively. We allow for different assets to have different scales for the idiosyncratic noise; the differences in β_i automatically imply differences

[4] For the zero mean normal distribution, the first order approximation to the tail probability is

$$\Pr\{X<-x\}\simeq\frac{1}{x}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\frac{x^2}{\sigma^2}\right).$$

Thus, changes in σ affect both the scale and the power.

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in scale for the market factor part. Thus, for large s

$$\Pr\{q_i \le -s\}/A_i = \Pr\{r_{mkt} \le -s\}/C = s^{-\alpha}[1+o(1)], \tag{18}$$

where $\alpha > 0$, and $A_i, C > 0$ for all i. Furthermore, assume that the β_i are uniformly distributed on the support $[a,b]$ and we restrict the analysis to $a > 0$ for simplicity of presentation. By repeatedly using the Feller convolution result, the return of an equally weighted n-stock portfolio $r(n)$ satisfies

$$\Pr\{r(n) \le -s\} = \left(CE[\overline{\beta}^{\alpha}] + n^{-\alpha}\sum_{i=1}^{n} A_i \right) s^{-\alpha}[1+o(1)] \tag{19}$$

as $s \to \infty$. Note that the assumption $\beta_i \in [a,b]$ implies that all moments are bounded. But the other random variables q_i and r_{mkt} only have moments up to α. After inversion, we get in analogy with (8)

$$\mathrm{VaR}_n = (CE[\overline{\beta}^{\alpha}] + n^{1-\alpha}\overline{A})^{1/\alpha}\delta^{-1/\alpha}[1+o(1)] \tag{20}$$

as $\delta \to 0$ and where $\overline{A} = (1/n)\sum_{i=1}^{n} A_i$.

The first term $E[\overline{\beta}^{\alpha}]$ can be expanded about the mean β to obtain[5]

$$\mathrm{VaR}_n = [\nu_1 + \nu_2 n^{-1} + \overline{\nu}_3 n^{1-\alpha} + O(n^{-2})]^{1/\alpha}[1+o(1)]. \tag{21}$$

Here $\nu_1 = C\beta^{\alpha}/\delta$, $\nu_2 = \frac{1}{2}C\beta^{\alpha-2}W\sigma_{\beta}^2/\delta$ and $\overline{\nu}_3 = \overline{A}/\delta$, assuming that $\overline{A}$ does not diverge as n increases. The coefficient $W \in [\alpha, k(k-1)]$ and where k is an integer closest to α such that $k \ge \alpha$. The factor W stems from the lower and upper bound of the Taylor expansion of the random variable β_i (random from the viewpoint of the random stockpicker); the bounds arise from the convexity of $(\overline{\beta})^{\alpha}$. Note that for $\alpha > 2$ the idiosyncratic part $\overline{\nu}_3 n^{1-\alpha}$ is of smaller order than the market factor term $\nu_2 n^{-1}$. As n increases these two factors determine the diversification effect for the VaR.

We can now obtain an explicit expression for the benefit of diversification for the mean-VaR safety-first investors if asset returns are heavy tailed distributed and when the acceptable risk level δ is low. Specifically, for the benefits $B_n \equiv R_{n^*} - R$, combining (5) and (21) gives

$$B_n^{VaR} \simeq \left(\frac{[\nu_1 + \nu_2 n^{-1} + \overline{\nu}_3 n^{1-\alpha}]^{1/\alpha} + R_f}{[\nu_1 + \nu_2 m^{-1} + \overline{\nu}_3 m^{1-\alpha}]^{1/\alpha} + R_f} - 1 \right) \times EP. \tag{22}$$

The RD of B_n^{VaR} follows as

$$\mathrm{RD}\{B_n^{VaR}\} = \left(\frac{EP}{\mathrm{VaR}_m + R_f} \right) \frac{1}{\alpha} \mathrm{VaR}_n^{1-\alpha}[-\nu_2 n^{-2} + (1-\alpha)\overline{\nu}_3 n^{-\alpha}]. \tag{23}$$

Note that there are two factors driving the $\mathrm{RD}\{B_n^{VaR}\}$ at different rates.

Compare (23) to (17) as we did in (13). Substituting (21) in (23) for VaR_n and (15) in (17) for σ_n, yields at larger values of n

$$\frac{\mathrm{RD}\{B_n^{VaR}\}}{\mathrm{RD}\{B_n^{stdv}\}} \simeq \frac{2\sigma_m \left(\frac{1}{\mathrm{VaR}_m + R_f} \right) \frac{1}{\alpha}}{(\sigma_{mkt}^2 \sigma_{\beta}^2 + R^2 \sigma_{\beta}^2 + \overline{\sigma}_q^2)} \beta \sigma_{mkt} (\nu_1)^{1/\alpha - 1}[\nu_2 + (\alpha-1)\overline{\nu}_3 n^{2-\alpha}] = c_m[\nu_2 + (\alpha-1)\overline{\nu}_3 n^{2-\alpha}], \tag{24}$$

say. One can see from (24) that in the end the ratio is positive, since the market factor term $\nu_2 n^{-1}$ in (21) disappears at a slower rate than the part due to idiosyncratic $\overline{\nu}_3 n^{1-\alpha}$ in (21) and hence the result in (24) if we divide by the rate n^{-2} stemming from the $\mathrm{RD}\{B_n^{stdv}\}$. Note that the case of multiple factors follows analogously. The question of diversification, though, is not what happens as $n \to \infty$, but what happens at moderate to larger values of $n < m$. In these cases not only the powers -2 and $-\alpha$ from (23) and -2 from (17) play a role, but also the constants ν_2 and $(1-\alpha)\overline{\nu}_3$ as well as the other elements in the ratio (24). To arrive at an answer to this question we employ simulations and conduct a calibration exercise.

3.3. *The normal distribution case*

We compare the diversification benefits of alternative risk measures under the assumption that returns are normally distributed. The case of the mean–variance optimizing investor in the single factor model is already covered by (16) above. Under the assumption that both the mean and variance exist, the analysis for the mean–variance investor is identical for return distributions with heavy tails (as long as $\alpha > 2$) and the normal distribution. However, the analysis for a safety-first type investor is somewhat different.

[5] For the detailed proof of the different scales A_i case, one need a minor changes of the proof in Hyung and de Vries (2010) where $A_i = A$ for all i is considered.

The standard deviation of the return on an equally weighted n-stock portfolio $r(n)$ can be expressed as in (15). Under normality, the normal based VaR level of $r(n)$, denoted as VaR_n^N, is given by

$$\text{VaR}_n^N = -R + z_\delta \sigma_n,$$

where $\Pr\{r(n) \le -\text{VaR}_n^N\} = \delta$, $r(n) \sim N(R, \sigma_n^2)$ and z_δ is the quantile of standard normal distribution such that $\delta < 0.5$. From (5) the incremental benefit of diversification to the safety-first investor who uses the VaR risk measure and given the normally distributed returns, is (the superscript refers to the normal distribution based VaR)

$$B_n^{NVaR} = \left(\frac{z_\delta \sigma_n - EP}{z_\delta \sigma_m - EP} - 1 \right) \times EP. \tag{25}$$

Upon differentiation we obtain the RD of B_n^{NVaR}

$$\text{RD}\{B_n^{NVaR}\} = -\frac{1}{2} \frac{z_\delta EP}{z_\delta \sigma_m - EP} \frac{1}{\sigma_n} (\sigma_{mkt}^2 \sigma_\beta^2 + R^2 \sigma_\beta^2 + \overline{\sigma}_q^2) n^{-2}. \tag{26}$$

Compare this rate with the standard deviation based RD (17)

$$\frac{\text{RD}\{B_n^{NVaR}\}}{\text{RD}\{B_n^{stdv}\}} = \frac{-\frac{1}{2} \frac{z_\delta EP}{z_\delta \sigma_m - EP} \frac{1}{\sigma_n} (\sigma_{mkt}^2 \sigma_\beta^2 + R^2 \sigma_\beta^2 + \overline{\sigma}_q^2) n^{-2}}{-\frac{1}{2} \frac{EP}{\sigma_m \sigma_n} (\sigma_{mkt}^2 \sigma_\beta^2 + R^2 \sigma_\beta^2 + \overline{\sigma}_q^2) n^{-2}} = \frac{z_\delta EP}{z_\delta \sigma_m - EP} \sigma_m. \tag{27}$$

As the ratio (27) is independent of n, this immediately shows that the two measures $\text{RD}\{B_n^{stdv}\}$ and $\text{RD}\{B_n^{NVaR}\}$ have the same RD.

4. Numerical calibration study

In the previous section, we compared the incremental benefits from diversification for different risk measures and under different assumptions regarding the distribution of returns. In the case of factor models, when returns are cross-sectionally dependent, there is more than one rate driving the diversification benefits. In those cases it is not only the power in the exponent (which determines the rate) that is important, but the scaling constants are important as well for the determination of the diversification benefits at moderate portfolio sizes n. In this section we employ a numerical calibration to gauge the relative importance of the factors contributing toward the benefits of diversification. We compare the diversification rates by calibrating the theoretical expressions by means of estimates of the parameters available in the relevant literature. In this section we consider only the case of identical distributions for the idiosyncratic risk. The heterogeneous idiosyncratic risk case is more readily studied by simulation, which is the topic of the next section.

For the costs of diversification, we use Statman's (2004) estimate of 0.06% additional net cost when moving from a small n-stock portfolio to the fully diversified portfolio. These additional net costs are assumed to be constant. The cost of holding the fully diversified portfolio is approximated by the expense ratio of the Vanguard Total Stock Market Index Fund, which at the time amounted to 0.20% per annum. Furthermore, Statman used 0.14% as a conservative estimate of the expected annual costs of buying and holding portfolios of individual stocks. The difference between these two estimates then yields the imputed 0.06% incremental costs.

To calculate the benefit of diversification, we further employ the following estimates by Statman's (2004) for comparability. The equity premium EP is set at 3.44%, the imputed size of the universe of assets m is 3444 (which is the number of stocks in the Vanguard Total Stock Market Index fund in March 2002), and the correlation between any pair of stocks is taken to be 0.08.[6] The risk-free rate is set at 2.19%, which is the estimate of Campbell et al. (2001). To calculate the benefits in the mean-VaR framework for the heavy tail case, we set the tail parameter α in (6) equal to 3 on the basis of Jansen and de Vries (1991). We equate all values for $A_i = A$ in (18) and set $A = 48.5$. This assumption of homogeneous scale is relaxed in the following section. Since the scale of the market factor differs profoundly from the scale of the idiosyncratic risk, we take $C = 5.7$; which is an estimate of the scale of the S&P 500 index as a representation for the market factor in the dataset of Hyung and de Vries (2010).[7] Furthermore, we used $\beta = 1$, $\sigma_\beta^2 = 1/12$ assuming that β_i's are spread evenly between 0.5 and 1.5. Furthermore, we focus on the tail probability $\delta = 0.001$, which amounts to approximately one event per five years.

[6] As one of the referees has pointed out, the average correlation among securities typically increases during bear markets. This issue is most readily addressed in the simulation study in the next section.

[7] The estimate of the variance of market factor is 7.1, which is significantly lower than the median value of the variance of the 1313 individual return series, i.e. 33.5, and the variance of the estimated idiosyncratic component, 29.9. We estimate the idiosyncratic components using the regression for the single index model. In the case of scale estimates, we find that the scale of the market factor is 5.7. Again this scale is substantially lower than the median values, respectively, 54.3 and 52.4, of the individual return series and the idiosyncratic components. One shows that these variances and scales are broadly consistent in the case that the market and idiosyncratic factor are assumed to follow Student's t distributions. The Appendix provides summary statistics.

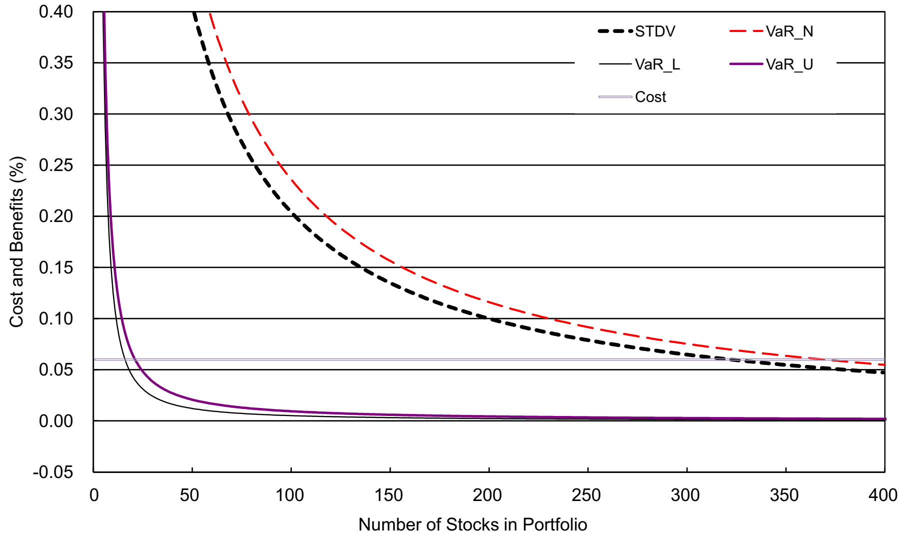

Fig. 2. Benefits of diversification: numerical calibration.

4.1. *Risk measures*

Fig. 2 shows the diversification benefits in terms of the VaR and the standard deviation σ_n under the assumption that the tail of the return distribution has heavy tails but finite variance. Given that $\alpha = 3$, which ensures the existence of the second moment of the return distribution, the variance (15) is invariant to the assumption regarding the distribution. Thus, we find that the dotted line (STDV) in Fig. 2, which gives the benefits from diversification for the mean–variance investor (16), is a replication of Statman's (2004) analysis, as we imputed exactly the same numerical calibrations. Considering the costs, the optimal level of diversification for the mean–variance investor is to hold approximately 300 different stocks.

The solid lines (VaR_L, VaR_U) in Fig. 2 represent the benefits from the decline in the VaR, calculated from (22). As shown in the previous section in (21), the VaR level is in between a lower and upper bound due to the variation in the betas. For $\alpha = 3$, the factor W in (21) turns out to lie within the interval [3, 6]. This implies that the optimal level of diversification ranges between 16 and 21 stocks. This range is considerably lower than the optimal level for the mean–variance type investor.

The benefits for the safety-first type investor come in two parts that have two different rates, see (22) and (23). The slow part with rate n^{-1} is due to the variability in beta and is equal to the rate at which the benefits come for the mean–variance investor, c.f. (16). The other part vanishes at rate $n^{1-\alpha}$, which implies a faster decline as long as $\alpha > 2$. Thus, the more rapid decline in the benefits from diversification for safety-first types stem from the differences in the rates due to the fatness of the tails.

4.2. *Equivalence under normality*

Finally, we demonstrate that two different risk measures are almost equivalent in terms of diversification benefits if we assume the asset returns to be normally distributed. The bold dotted line (STDV) in Fig. 2 representing (16) is again a replication of Statman's (2004) mean–variance analysis under the normality assumption. The benefits of diversification in terms of mean-VaR under the normality assumption are provided in (25). The dashed line (VaR_N) in Fig. 2 illustrates the incremental cost and benefit of increasing diversification for the mean-VaR investor. The risk reduction benefit of increasing diversification beyond 300 stocks is less than the net cost of 0.06%, yielding approximately 300 different stocks in the optimal portfolio. The VaR_N (dashed) line represents the decline in VaR calculated from (25) under the normality assumption. The optimal level is to hold 369 stocks for the mean-VaR investor. We conclude that there is no substantive qualitative difference between two different risk measures under the assumption of normality.

5. Simulation study

In this section we consider the case of heterogeneous idiosyncratic risk and varying correlation. The effects of heterogeneity and varying correlation are most easily studied by simulation. The design of the simulations is as follows. In this simulation study we use the same parameter values as in the calibration exercise of Section 4, except for the scale parameters, c.f. (18). The returns r_i derive from a factor model (14) in which the innovations are i.i.d. with a three

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parameter version of the Student's t distribution so that the variances can differ, even though the degrees of freedom are constant.

In particular we generate t-distributed random numbers with mean μ, with a scale parameter ξ,

$$f(y|\mu,\xi,v) = \frac{\Gamma((v+1)/2)}{\xi\Gamma(1/2)\Gamma(v/2)v^{1/2}}\left[1+\frac{(y-\mu)^2}{v\xi^2}\right]^{-(v+1)/2},$$

where $y=(-\infty,+\infty)$, $\mu=(-\infty,+\infty)$, $\sigma>0$ and degrees of freedom $v>0$. It has standard deviation

$$\xi[v/(v-2)]^{1/2}. \tag{28}$$

One shows in (6) that $\alpha = v$, since by l'Hopital's rule

$$\lim_{t\to\infty}\frac{f(tx)}{f(t)} = x^{-v-1}.$$

The scale A can be isolated by

$$\lim_{y\to\infty}\frac{f(y)}{vy^{-v-1}} = \frac{\Gamma((v+1)/2)}{\Gamma(1/2)\Gamma(v/2)}v^{(v/2-1)}\xi^{v} = A. \tag{29}$$

So the variance factor ξ^v indicates by how much A changes in comparison with the standard two parameter Student distribution case.

As before, the stock beta's, β_i, are assumed to be evenly spread between 0.5 and 1.5. We assume that the (excess) return on the market portfolio r_{mkt} follows a Student's t distribution with mean $\mu=0.0344$ (i.e. amounting to 3.44% yearly return), scale parameter $\xi=1$, and has degrees of freedom $v=3$. This means that the tail index of r_{mkt} is $\alpha=3$.

The idiosyncratic term q_i is independent with respect to r_{mkt}, and is distributed with a Student's t distribution with location parameter $\mu=0$ and the degree of freedom $v=3$. Thus, the market part and the idiosyncratic noise have the same tail index. We simulate both with a homogenous scale parameter for comparability with the previous section (i.e. $A_i = A$ for all i) and with heterogenous scale parameters A_i, as in (18). The different scale values A_i are induced by selecting different values for the scale parameter ξ of the Student distribution. More specifically, we use

(1) Homogeneous case: $\xi=3$;
(2) Heterogeneous case: draw ξ from a Chi-square distribution with the degrees of freedom 4.5.

We simulate 1000 different artificial stock return series. Each series comes with 10,000 observations. Table A1 in the Appendix shows that the properties of the simulated series come close to what is observed for the actual stock returns that were used by Hyung and de Vries (2010). For example, the correlation coefficients of the 'actual series' column reported in the row of 'corr. coef.' in Table A1 are the averaged values from all pairs of 1313 stock returns. Additionally, we find that the value of 0.08 is exactly the same as Statman's (2004) correlation between any pair of stocks.

We also simulated data to generate different values of correlation structure between the stocks. The expected correlation of Statman (2004), which was based on five years of monthly data as elucidated by Campbell et al. (2001), was 0.08. However, Campbell et al. (2001) claimed that correlations based on one year of daily data decline from 0.12 in the early 1960s to between 0.02 and 0.04 in the 1990s. The sample period in the study of Hyung and de Vries (2010) runs from 1985 to 2005, which covers a span of 30 years of daily data. For the IT bubble period following the year 2000, we estimated the average correlation and find a higher correlation of around 0.11. As a robustness check, we also simulated using 0.04 for the low correlation situation and 0.11 for the high correlation case. To achieve this, we set different scale parameters (ξ) for the idiosyncratic term q_i, which leads to differences in the variances as in (28). For the given value of the variances of r_{mkt}, since the correlation between stocks is inversely related to the variances of q_i, the larger (smaller) ξ of the idiosyncratic term, the lower (higher) is the correlation. Furthermore, in the heterogeneous case, we draw ξ from the Chi-square distribution. If we set a larger (smaller) degree of freedom for the Chi-square distribution, the induced correlation between simulated series is lower (higher). In particular, we select

(3) For the homogeneous case take $\xi=2.5$ and 4.5.
(4) For the distribution of ξ in the heterogeneous case we use, respectively, 4.0 and 6.0 degrees of freedom.

We construct equally weighted n-stock portfolio returns $r(n)$ by randomly selecting n stocks from the 1000 assets. From the empirical distribution of n-stock portfolio returns we calculate the standard deviation and empirical VaR. Each experiment is conducted with 500 replications. Thus, the averages of the standard deviation, artificial historical VaR from 500 different naive-portfolios with n-stocks are calculated for each $n=1,2,\ldots$. Subsequently, we calculate the corresponding incremental benefits from diversification as per formulas (2) and (5) at risk levels of $\delta=0.05$, 0.01, 0.005 and 0.001 risk levels. Recall that an event with probability $\delta=0.001$ at the daily frequency corresponds to an extreme event that may occur about once every five years. The δ-level 0.05 reflects events that occur approximately every month. So for investors with a genuine concern for downside risk, only the δ-levels below 5% are relevant. The optimal level of diversification depends on where these incremental benefits equate with the incremental costs. To this end, we again use

Statman's (2004) estimate of 0.06% additional net cost when moving from a small n-stock portfolio to the fully diversified portfolio. We use an equity premium of 3.44% and the risk-free rate is 2.19%, as before.

5.1. Homogeneous scale

We begin with the case of a homogenous scale parameter for the idiosyncratic noise with $\xi = 3$. The results in Table 1 demonstrate that, in the case of the mean–variance criterion, the optimal level of diversification is approximately 250 stocks, which is close to the level of 300 reported in Section 4. For the mean-VaR investor, the optimal level of diversification is 200 stocks if the risk level is $\delta = 0.05$, while this declines to a mere 14 stocks at $\delta = 0.001$. The STDV_homo and VaR01_homo lines in Fig. 3 represent the benefits from declines in standard deviations and VaR at $\delta = 0.001$. Thus, one needs to assume that the downside risk concern is really about tail events to be able to replicate the observed low levels of diversification in actual portfolios of individuals.

5.2. Heterogeneous scale

Next, consider the case of heterogenous scales for the idiosyncratic noise part. The results in Table 1 show that in the case of the mean–variance investor the optimal level of diversification is about 800 stocks. This is larger than the optimal level for the homogenous case, which is approximately 300 stocks. Thus, the incremental benefits of diversification come rather slowly. For the mean-VaR investor, the optimal level of diversification is 700 stocks at the risk level of $\delta = 0.05$. This declines to a mere 65 stocks at $\delta = 0.001$. Two lines (STDV_hetero, VaR01_hetero) in Fig. 3 demonstrate this graphically. The simulation experiments demonstrate that heterogeneity considerably attenuates the speed of the diversification benefits, for both the mean–variance investor and the safety-first type investors. Moreover one notices that at moderate risk levels such as the 0.05 δ-level, the portfolio size is considerable under all criteria. Third, deeper into the tails at the

Table 1
Excess benefits of diversification.

#	Homogeneous case					Heterogeneous case				
	STDV	VaR5	VaR1	VaR05	VaR01	STDV	VaR5	VaR1	VaR05	VaR01
2	4.76	2.69	2.95	2.78	2.49	9.32	4.96	5.93	5.76	6.11
4	2.82	1.72	1.66	1.44	0.98	6.09	3.44	3.79	3.49	3.42
6	2.04	1.30	1.15	0.93	0.47	4.64	2.70	2.82	2.50	2.26
8	1.59	1.03	0.85	0.66	0.24	3.77	2.25	2.25	1.93	1.66
10	1.32	0.88	0.68	0.51	0.14	3.20	1.94	1.88	1.57	1.20
12	1.10	0.74	0.53	0.38	0.06	2.77	1.71	1.61	1.29	0.93
14	0.96	0.65	0.46	0.31	**0.03**	2.59	1.62	1.50	1.19	0.82
16	0.84	0.57	0.38	0.24	0.00	2.23	1.41	1.26	0.97	0.57
18	0.76	0.53	0.33	0.21	−0.01	2.09	1.33	1.18	0.89	0.51
20	0.69	0.47	0.29	0.18	−0.02	1.85	1.19	1.02	0.75	0.39
25	0.55	0.38	0.21	0.11	−0.04	1.56	1.02	0.85	0.59	0.26
30	0.44	0.31	0.15	0.07	−0.07	1.33	0.89	0.70	0.46	0.16
35	0.39	0.27	0.12	0.05	−0.06	1.15	0.77	0.58	0.36	0.12
40	0.32	0.22	0.09	0.02	−0.07	1.02	0.69	0.51	0.30	0.07
45	0.29	0.20	0.08	**0.02**	−0.07	0.93	0.63	0.45	0.26	0.07
50	0.25	0.17	0.06	0.00	−0.08	0.82	0.56	0.39	0.20	0.03
55	0.21	0.15	0.04	−0.02	−0.09	0.74	0.51	0.34	0.16	0.02
60	0.19	0.13	0.03	−0.02	−0.08	0.68	0.47	0.31	0.14	0.00
65	0.19	0.13	0.03	−0.02	−0.08	0.64	0.45	0.28	0.12	**0.00**
70	0.16	0.11	0.02	−0.04	−0.08	0.60	0.42	0.26	0.10	−0.01
75	0.15	0.10	**0.02**	−0.04	−0.08	0.55	0.38	0.23	0.08	−0.01
100	0.09	0.06	−0.01	−0.06	−0.09	0.40	0.28	0.15	0.03	−0.04
125	0.08	0.05	−0.01	−0.05	−0.07	0.33	0.23	0.12	**0.01**	−0.02
150	0.04	0.02	−0.03	−0.06	−0.08	0.25	0.18	0.08	−0.02	−0.04
200	0.02	**0.00**	−0.03	−0.07	−0.08	0.18	0.13	0.04	−0.03	−0.04
250	**0.01**	−0.01	−0.04	−0.07	−0.08	0.14	0.10	0.03	−0.05	−0.05
300	0.00	−0.02	−0.05	−0.07	−0.08	0.10	0.07	0.01	−0.05	−0.05
350	−0.01	−0.02	−0.06	−0.07	−0.09	0.08	0.05	**0.00**	−0.06	−0.06
400	−0.02	−0.03	−0.05	−0.07	−0.08	0.07	0.04	0.00	−0.06	−0.06
500	−0.03	−0.03	−0.06	−0.07	−0.09	0.04	0.02	−0.02	−0.07	−0.06
600	−0.03	−0.04	−0.06	−0.07	−0.08	0.03	0.01	−0.02	−0.07	−0.06
700	−0.04	−0.04	−0.06	−0.07	−0.08	0.01	**0.00**	−0.03	−0.07	−0.07
800	−0.04	−0.04	−0.07	−0.07	−0.08	**0.00**	0.00	−0.03	−0.07	−0.06
1000	−0.06	−0.06	−0.06	−0.06	−0.06	−0.06	−0.06	−0.06	−0.06	−0.06

Note: STDV and VaR denote standard deviation and value-at-risk. VaR is calculated at the following probabilities: 0.05, 0.01, 0.005 and 0.001. We set an equity premium 3.44%, the risk-free rate 2.19% and additional net cost 0.06%. Boldface figures emphasize the optimal level of selected risk-diversification w.r.t. the number of stocks.

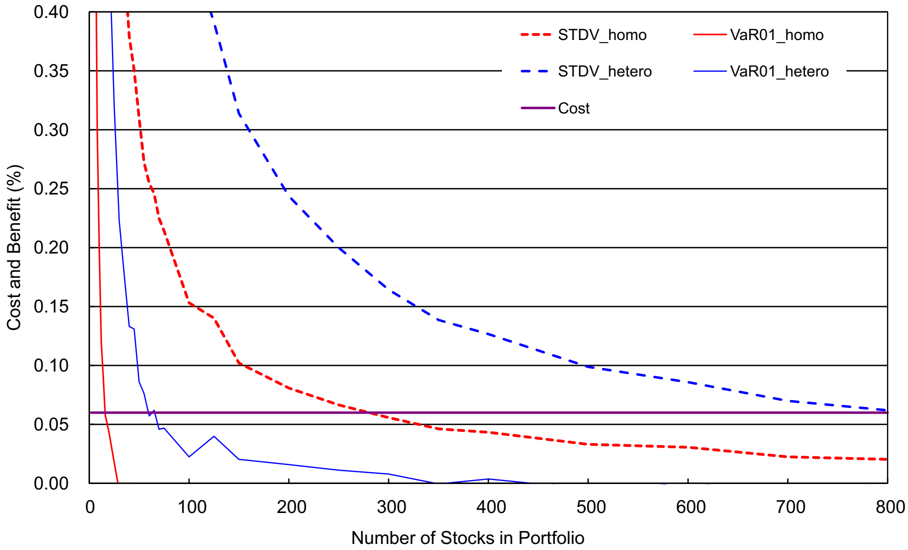

Fig. 3. Benefits of diversification using simulated data.

Table 2
Optimal level of diversification (robustness check).

	Corr. coef.	STDV	VaR5	VaR1	VaR05	VaR01
ξ of Student's *t*	Homogeneous case					
2.5	0.128	175	175	55	35	10
4.5	0.044	600	500	250	100	35
dof of Chi-square	Heterogeneous case					
4.0	0.117	600	600	250	125	40
6.0	0.048	900	900	500	200	65

Note: For the homogeneous case the idiosyncratic noise is drawn from Student's *t* distribution with scale parameter $\xi = 2.5$ and 4.5. In the heterogeneous case, the scale parameters ξ come from a Chi-square distribution with the degree of freedom (dof) 4.0 and 6.0. Corr. coef. denotes the averaged value of correlation coefficients from simulated series.

more extreme risk levels, there is once again a large difference in the amount of diversification between the mean–variance investor and the safety-first investor who relies on the mean-VaR criterion.

5.3. *Robustness check*

Since the correlation between stocks varies across time we investigate the effects of different levels of correlation. This also provides for a robustness check. The results of these simulations are shown in Table 2. First, consider the series with low correlation. We set $\xi = 4.5$ for the homogeneous case; this induces an average value of correlation of 0.044, which is equal to the 'corr. coef.' row of Table A1. We find that the optimal level of diversification by the mean–variance investor entails holding 600 stocks, for the mean-VaR investor with $\delta = 0.05$ the number is 500 stocks, while it is only 35 stocks for the mean-VaR investor with $\delta = 0.001$. In the case of heterogenous scales for the idiosyncratic noise part we draw ξ from a Chi-square distribution with the degrees of freedom 6.0. The averaged value of correlation coefficients is 0.048. We find that the optimal levels of diversification by the mean–variance, and by the mean-VaR with $\delta = 0.05$ are over 900, while it is only 65 stocks for the mean-VaR at $\delta = 0.001$.

To induce a higher level of correlation, we set $\xi = 2.5$ for the homogeneous case; this induces an average value of correlation of 0.128. For the heterogenous case with higher correlation, we draw ξ from a Chi-square distribution with the degree of freedom 4.0, where the averaged value of correlation is 0.117. We find that the optimal level of diversification by the mean-VaR with $\delta = 0.001$ entails holding 10 stocks in the homogeneous case, while it is 40 stocks in the heterogenous case.

In the simulation the lower (higher) correlation for the heterogeneous case is induced by higher (lower) value of the scale $\overline{A}$, which determines the coefficient $\overline{v}_3$ of the part that vanishes at the rate $n^{1-\alpha}$ in (23). The rate of decline in the benefits from diversification is unaffected by changes in the scale $\overline{A}$, as this does not affect the power. However, since the coefficient $\overline{v}_3 = \overline{A}/\delta$ becomes larger with increasing value of $\overline{A}$, one selects more stocks to reach the optimal diversification level at each desired δ-level. This also applies in the case of homogeneous scale. As the correlation is reduced, the number of stocks in a well-diversified portfolio increases by any criterion such as mean–variance or mean-VaR (in the case of perfect correlation, only a single stock would suffice).

6. Conclusion

The diversification puzzle is that actual portfolios of individuals contain a much smaller amount of different stocks than mean–variance theory would prescribe. The calibration and simulation studies demonstrate that the concern over downside risk at a sufficiently low probability level combined with the fat tail phenomenon can be used to replicate the low diversification phenomenon. Our companion paper, Hyung and de Vries (2010), furnishes empirical evidence for this finding. The fat tail phenomenon is now well recognized and popularized as the black swan. The concern for downside risk is at least institutionalized in the financial industry through VaR restrictions; behavioral economics has shown the importance of loss aversion. What we take away from the above investigation is that in some form the downside risk criterion may be necessary to recognize the fat tail phenomenon. The combination leads to a relatively lower level of optimal portfolio sizes.

It may sound counter-intuitive that a concern for downside leads to lower diversification, given the fat tail phenomenon. This confusion arises from the fact that the fat tail phenomenon is a comparison of the normal distribution based tail and the tails of distributions such as the Student distribution. This comparison is done as the probability level is driven to zero. The diversification comparison is a cross-sectional comparison whereby the portfolio size is varied but the probability level is held constant. Cross-sectional aggregation under normality just works just too well. Diversification reduces the variance and thereby affects the power. But for fat tailed distributions, only the scale is affected, not the power. The rate effect of the normal dominates over the scale effect for the heavy tailed distributions.

Our research calls for further investigation on optimal selection of n stocks among m. DeMiguel et al. (2009) for example suggest the out-of-sample comparison between naive portfolio versus various asset allocation models.

Acknowledgments

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Appendix A. Details of the data

In this appendix we provide details of the actual data used by Hyung and de Vries (2010) and details about how to generate simulated data. The dataset in Hyung and de Vries (2010) includes 888 stocks from the NYSE and 425 stocks from the NASDAQ (a total 1313 stocks), close-to-close daily returns including cash dividends. The sample period is from January 1, 1985 to

Table A1
Actual and simulated series: summary of statistics.

Empirical statistics	Actual series	Heterogeneous case			Homogeneous case		
		dof=4.0	dof=4.5	dof=6.0	$\xi=2.5$	$\xi=3.0$	$\xi=4.5$
Std. dev.							
Min	1.82	1.03	1.03	1.43			
25th percentile	4.49	3.71	4.30	6.23			
Median	5.79	5.95	6.88	9.33	4.61	5.40	7.85
75th percentile	8.43	9.07	10.41	13.33			
Max	38.10	34.44	35.97	35.85			
A							
Min	0.26	0.21	0.23	0.58			
25th percentile	17.50	10.08	15.75	47.90			
Median	54.20	41.53	64.25	160.54	19.35	31.20	95.66
75th percentile	252.70	147.64	223.05	468.10			
Max	986,351.30	8070.87	9195.04	9104.67			
Corr. coef.	0.080	0.117	0.086	0.048	0.128	0.093	0.044

Note: Std. dev., *A*, and corr. coef. denote, respectively, the standard deviation, the scale in the Pareto expansion of the distribution, and the averaged value of correlation coefficients. The 'actual series' column reports empirical statistics for 1313 stock returns in Hyung and de Vries (2010); see the Appendix for detailed information on the data. The columns with headings 'heterogeneous case' and 'homogeneous case' report empirical statistics for 1000 simulated series. The values in the column of std. dev. and *A* are quantiles. (We report median only in the 'homogeneous case'.) The values in the row of corr. coef. are averaged values. The value 'dof' in the columns of 'heterogeneous case' denotes the degrees of freedom of Chi-square distribution used to generate scale parameters for the Student's *t*-distribution. The 'ξ' in the columns of the 'homogeneous case' denotes the scale parameters for the Student's *t*-distribution.

February 15, 2005 giving a sample size of 5251. We generate artificial stock return series to replicate the heterogeneous scale properties of the actual series using the three parameter Student's t distribution. For the comparison with the calibration exercise, we also generate data with a homogeneous scale.

The summary of statistics of the 1313 stocks in Hyung and de Vries (2010) are quite closely replicated by the simulated series, see the summary of statistics in Table A1. It turns out that the histogram of estimated standard errors of the 1313 stocks is skewed to the right, with median 5.79. The median value of the scale parameters is 54.2. To generate series with heterogenous scales, we draw scale parameter ξ from a Chi-square distribution with degrees of freedom 4.0, 4.5 and 6.0. Note that the mean value of the Chi-square distribution equals the degree of freedom. For the homogeneous case, we choose ξ to be 2.5, 3.0 and 4.5, respectively. The $\xi = 3$ value provides a good approximation to the actual data.

The correlation coefficients reported in the 'corr. coef.' row in Table A1 are average of correlation coefficients computed for all pairs of 1313 actual returns or 1000 simulated series. In the panels of 'std. dev.' and 'A', we report the minimum, 25th percentile, median, 75th percentile and maximum out of 1000 simulated series or 1313 stock returns, respectively.

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