A General Purpose Technology Explains the Solow Paradox and Wage Inequality: Appendix

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Abstract

This note contains the unpublished appendix of Bas Jacobs and Richard Nahuis (2002), “A General Purpose Technology Explains the Solow Paradox and Wage Inequality”, Economics Letters, 74, 243-250. The appendix contains derivations of i) the first-order conditions, ii) equilibrium of the model, iii) the conditions for stability, and iv) the derivation of the slopes of the phase lines.

First-order conditions

Firms maximize the discounted value of profits flows \( \Pi_j \equiv \int_0^\infty \pi_j \exp[-rt] \, dt \), subject to the demand function for their variety, \( X_j = \left( \frac{p_j}{p_X} \right)^{-\varepsilon} X \), and the technology accumulation constraint given in equation (3) in the text. Instantaneous profits are given by: \( \pi_j \equiv p_j X_j - w_L L_j - w_H H_j - r K_j \). Therefore, the current-value Hamiltonian of the optimal control problem for firm \( j \) reads as:

\[
H_j = p_j X_j - w_H H_j - w_L L_j - r K_j + q_j B(1 - u_j) H_j F_j
\]

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Define $\theta \equiv (1-\alpha)\beta$ and $\xi \equiv (1-\alpha)(1-\beta)$. First-order conditions (FOC’s) for an optimum are:

$$\frac{\partial H_j}{\partial L_j} = p_j \frac{\varepsilon - 1}{\varepsilon} \xi A K_j^\alpha F_j^{1-\alpha}(u_j H_j)^{\theta} L_j^{\xi - 1} - w_L = 0$$

$$\frac{\partial H_j}{\partial H_j} = p_j \frac{\varepsilon - 1}{\varepsilon} \theta A K_j^\alpha F_j^{1-\alpha}(u_j H_j)^{\theta} H_j^{-1} L_j^{\xi} - w_H + q_j B(1-u_j)F_j = 0$$

$$\frac{\partial H_j}{\partial u_j} = p_j \frac{\varepsilon - 1}{\varepsilon} \theta A K_j^\alpha F_j^{1-\alpha}(u_j H_j)^{\theta} u_j^{-1} L_j^{\xi} - q_j B H_j F_j = 0$$

$$\frac{\partial H_j}{\partial K_j} = p_j \frac{\varepsilon - 1}{\varepsilon} \alpha A K_j^\alpha L_j^{\xi} - r = 0$$

$$\frac{\partial H_j}{\partial F_j} = p_j \frac{\varepsilon - 1}{\varepsilon} (1-\alpha) A K_j^\alpha F_j^{-\alpha}(u_j H_j)^{\theta} L_j^{\xi} + q_j B(1-u_j)H_j = r q_j - \dot{q}_j$$

in addition to the transversality condition:

$$\lim_{t \to \infty} F_j \exp\left(-\int_0^t r(v)dv\right) = 0$$

**Equilibrium**

The second and third FOC’s give the no-arbitrage condition for the allocation of time of high-skilled workers in the production of goods and learning in the text.

The differential equation for $R \equiv F/K$ can be obtained using the economy’s resource constraint - after imposing symmetric equilibrium:

$$\frac{\dot{R}}{R} = B(1-u)H - spA R^{1-\alpha}(uH)^{\theta} L^{\xi}$$

The derivation of the differential equation describing $u$ requires two additional steps. First, the no-arbitrage condition of high-skilled workers can be differentiated with respect to time to arrive at:

$$\alpha \frac{\dot{R}}{R} + (1-\theta) \frac{\dot{u}}{u} = -\frac{\dot{q}}{q}$$
Second, we can substitute the first term in the last FOC out by rewriting the no-arbitrage condition:

\[(1 - \alpha)AR^{-\alpha}(uH)^\theta L^\xi = \frac{1 - \alpha}{\theta} qBuH\]

Using the last three results we obtain the differential equation for \(u\):

\[\frac{\dot{u}}{u} = \frac{1 - \alpha}{\theta} BuH + \frac{1 - \alpha}{1 - \theta} BH + \frac{\alpha(1 - sp)}{\theta - 1} AR^{1-\alpha}(uH)^\theta L^\xi\]

Equilibrium follows by setting \(\frac{\dot{u}}{u} = 0\) and \(\frac{\dot{R}}{R} = 0\) and solving for \(u^*\) and \(R^*\).

**Stability**

The stability of the equilibrium can be checked by evaluating the determinant of Jacobian matrix \(J\) at the equilibrium \(E\). The four partial derivatives of \(J\) at \(E\) are:

\[\frac{\partial \dot{R}}{\partial R} \bigg|_E = -(1 - \alpha)B(1 - u^*)H < 0\]

\[\frac{\partial \dot{R}}{\partial u} \bigg|_E = -BHR^*(1 + \theta(1 - u^*/u^*)) < 0\]

\[\frac{\partial \dot{u}}{\partial u} \bigg|_E = 2 \left(\frac{1 - \alpha}{\theta}\right) BHu^* + \left(\frac{1 - \alpha}{1 - \theta}\right) BH - (1 + \theta) \left(\frac{(1 - \alpha)}{\theta} BHu^* + \frac{(1 - \alpha)}{(1 - \theta)} BH\right)\]

\[= \frac{1 - \alpha}{\theta} BHu^* - \frac{\theta \phi}{u^*}\]

\[\frac{\partial \dot{u}}{\partial R} \bigg|_E = -(1 - sp) \left(\frac{1 - \alpha}{1 - \theta}\right) \frac{R^*u^*}{R^*} < 0\]

where \(\phi \equiv ((1 - \alpha)/\theta) BHu^* + ((1 - \alpha)/(1 - \theta)) BH\). The equilibrium is saddle-point stable if \(\partial \dot{u}/\partial u > 0\). Then, the determinant of the Jacobian is negative. This will be the case if:

\[\frac{1 - \alpha}{\theta} BHu^* - \frac{\theta \phi}{u^*} > 0\]

substitution of \(\phi\) gives:

\[u^* > \left(\frac{\theta}{1 - \theta}\right)^2\]
Slopes phase-lines

The slopes of the curves in figure 1 are derived by totally differentiating the $\dot{R} = 0$ and $\dot{u} = 0$ lines with respect to $R$ and $u$. The $\dot{R} = 0$ locus is downward sloping:

$$\frac{du}{dR}|_{\dot{R}=0} = -\frac{u(1-\alpha)/R}{\theta + u/(1-u)} < 0$$

The $\dot{u} = 0$ locus is upward sloping:

$$\frac{du}{dR}|_{\dot{u}=0} = \frac{(1-\alpha)\phi/R}{(1-\alpha)BH/\theta - \theta\phi/u} > 0$$

The denominator is positive as a consequence of the stability condition, i.e. when $u^* > ((\theta/(1-\theta))^2$. 
